CHAPTER 5
Logarithmic, Exponential, and Other Transcendental Functions

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Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Odd-Numbered Exercises

1. Simpson's Rule: $n = 10$

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x \frac{1}{t}\,dt$	-0.6932	0.4055	0.6932	0.9163	1.0987	1.2529	1.3865

Note: $\displaystyle\int_1^{0.5} \frac{1}{t}\,dt = -\int_{0.5}^{1} \frac{1}{t}\,dt$

3. (a) $\ln 45 \approx 3.8067$

(b) $\displaystyle\int_1^{45} \frac{1}{t}\,dt \approx 3.8067$

5. (a) $\ln 0.8 \approx -0.2231$

(b) $\displaystyle\int_1^{0.8} \frac{1}{t}\,dt \approx -0.2231$

7. $f(x) = \ln x + 2$

Vertical shift 2 units upward

Matches (b)

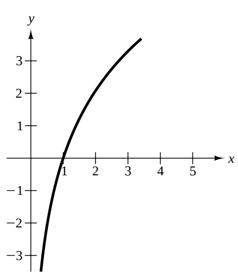
9. $f(x) = \ln(x - 1)$

Horizontal shift 1 unit to the right

Matches (a)

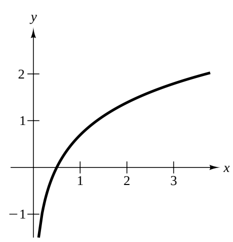
11. $f(x) = 3 \ln x$

Domain: $x > 0$



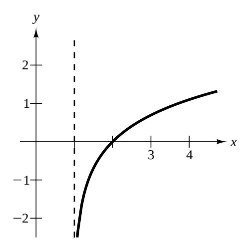
13. $f(x) = \ln 2x$

Domain: $x > 0$



15. $f(x) = \ln(x - 1)$

Domain: $x > 1$



17. (a) $\ln 6 = \ln 2 + \ln 3 \approx 1.7917$

(b) $\ln \frac{2}{3} = \ln 2 - \ln 3 \approx -0.4055$

(c) $\ln 81 = \ln 3^4 = 4 \ln 3 \approx 4.3944$

(d) $\ln \sqrt{3} = \ln 3^{1/2} = \frac{1}{2} \ln 3 \approx 0.5493$

19. $\ln \frac{2}{3} = \ln 2 - \ln 3$

21. $\displaystyle\ln \frac{xy}{z} = \ln x + \ln y - \ln z$

23. $\displaystyle\ln \sqrt[3]{a^2 + 1} = \ln(a^2 + 1)^{1/3} = \frac{1}{3} \ln(a^2 + 1)$

25. $\ln\left(\frac{x^2-1}{x^3}\right)^3 = 3[\ln(x^2-1) - \ln x^3]$

$= 3[\ln(x+1) + \ln(x-1) - 3\ln x]$

27. $\ln z(z-1)^2 = \ln z + \ln(z-1)^2$

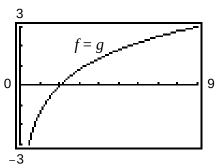
$= \ln z + 2\ln(z-1)$

29. $\ln(x-2) - \ln(x+2) = \ln\frac{x-2}{x+2}$

31. $\frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)] = \frac{1}{3}\ln\frac{x(x+3)^2}{x^2-1} = \ln\sqrt[3]{\frac{x(x+3)^2}{x^2-1}}$

33 . $2\ln 3 - \frac{1}{2}\ln(x^2+1) = \ln 9 - \ln\sqrt{x^2+1} = \ln\frac{9}{\sqrt{x^2+1}}$

35.



37. $\lim_{x \to 3^+} \ln(x-3) = -\infty$

39. $\lim_{x \to 2^-} \ln[x^2(3-x)] = \ln 4 \approx 1.3863$

41. $y = \ln x^3 = 3\ln x$

$y' = \frac{3}{x}$

At $(1, 0)$, $y' = 3$.

43. $y = \ln x^2 = 2\ln x$

$y' = \frac{2}{x}$

At $(1, 0)$, $y' = 2$.

45. $g(x) = \ln x^2 = 2\ln x$

$g'(x) = \frac{2}{x}$

47. $y = (\ln x)^4$

$\frac{dy}{dx} = 4(\ln x)^3\left(\frac{1}{x}\right) = \frac{4(\ln x)^3}{x}$

49. $y = \ln x\sqrt{x^2-1} = \ln x + \frac{1}{2}\ln(x^2-1)$

$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\left(\frac{2x}{x^2-1}\right) = \frac{2x^2-1}{x(x^2-1)}$

51. $f(x) = \ln\frac{x}{x^2+1} = \ln x - \ln(x^2+1)$

$f'(x) = \frac{1}{x} - \frac{2x}{x^2+1} = \frac{1-x^2}{x(x^2+1)}$

53. $g(t) = \frac{\ln t}{t^2}$

$g'(t) = \frac{t^2(1/t) - 2t\ln t}{t^4} = \frac{1-2\ln t}{t^3}$

55. $y = \ln(\ln x^2)$

$\frac{dy}{dx} = \frac{1}{\ln x^2}\frac{d}{dx}(\ln x^2) = \frac{(2x/x^2)}{\ln x^2} = \frac{2}{x\ln x^2} = \frac{1}{x\ln x}$

57. $y = \ln\sqrt{\frac{x+1}{x-1}} = \frac{1}{2}[\ln(x+1) - \ln(x-1)]$

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{1-x^2}$

59. $f(x) = \ln\frac{\sqrt{4+x^2}}{x} = \frac{1}{2}\ln(4+x^2) - \ln x$

$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} = \frac{-4}{x(x^2+4)}$

61. $y = \dfrac{-\sqrt{x^2+1}}{x} + \ln\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = \frac{-x\left(x/\sqrt{x^2+1}\right) + \sqrt{x^2+1}}{x^2} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x^2\sqrt{x^2+1}} + \left(\frac{1}{x+\sqrt{x^2+1}}\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = \frac{1}{x^2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} = \frac{1+x^2}{x^2\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x^2}$$

63. $y = \ln|\sin x|$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$$= \ln|\cos x| - \ln|\cos x - 1|$$

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} - \frac{-\sin x}{\cos x - 1} = -\tan x + \frac{\sin x}{\cos x - 1}$$

67. $y = \ln\left|\dfrac{-1+\sin x}{2+\sin x}\right|$

$$= \ln|-1+\sin x| - \ln|2+\sin x|$$

$$\frac{dy}{dx} = \frac{\cos x}{-1+\sin x} - \frac{\cos x}{2+\sin x}$$

$$= \frac{3\cos x}{(\sin x - 1)(\sin x + 2)}$$

69. $f(x) = \sin 2x \ln x^2 = 2\sin 2x \ln x$

$$f'(x) = (2\sin 2x)\left(\frac{1}{x}\right) + 4\cos 2x \ln x$$

$$= \frac{2}{x}(\sin 2x + 2x\cos 2x \ln x)$$

$$= \frac{2}{x}(\sin 2x + x\cos 2x \ln x^2)$$

71. (a) $y = 3x^2 - \ln x, \quad (1, 3)$

$$\frac{dy}{dx} = 6x - \frac{1}{x}$$

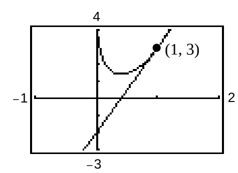
When $x = 1$, $\dfrac{dy}{dx} = 5$.

Tangent line: $y - 3 = 5(x - 1)$

$$y = 5x - 2$$

$$0 = 5x - y - 2$$

(b)



73. $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

75. $y = 2(\ln x) + 3$

$$y' = \frac{2}{x}$$

$$y'' = -\frac{2}{x^2}$$

$$xy'' + y' = x\left(-\frac{2}{x^2}\right) + \frac{2}{x} = 0$$

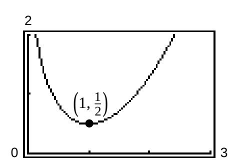
77. $y = \dfrac{x^2}{2} - \ln x$

Domain: $x > 0$

$$y' = x - \frac{1}{x} = \frac{(x+1)(x-1)}{x} = 0 \text{ when } x = 1.$$

$$y'' = 1 + \frac{1}{x^2} > 0$$

Relative minimum: $\left(1, \frac{1}{2}\right)$



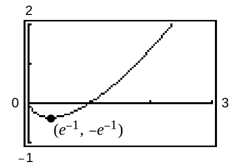
79. $y = x \ln x$

Domain: $x > 0$

$$y' = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x = 0 \text{ when } x = e^{-1}.$$

$$y'' = \frac{1}{x} > 0$$

Relative minimum: $(e^{-1}, -e^{-1})$



81. $y = \dfrac{x}{\ln x}$

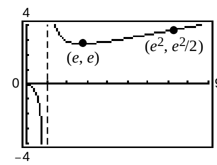
Domain: $0 < x < 1, x > 1$

$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{(\ln x)^2(1/x) - (\ln x - 1)(2/x)\ln x}{(\ln x)^4} = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $(e^2, e^2/2)$



83. $f(x) = \ln x, \quad f(1) = 0$

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1$$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1, \quad P_1(1) = 0$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

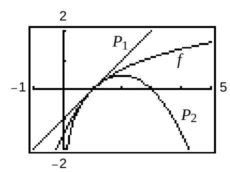
$$= (x - 1) - \frac{1}{2}(x - 1)^2, \quad P_2(1) = 0$$

$P_1'(x) = 1, \quad P_1'(1) = 1$

$P_2'(x) = 1 - (x - 1) = 2 - x, \quad P_2'(1) = 1$

$P_2''(x) = -1, \quad P_2''(1) = -1$

The values of f, P_1, P_2, and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



85. Find x such that $\ln x = -x$.

$$f(x) = (\ln x) + x = 0$$

$$f'(x) = \frac{1}{x} + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n\left[\frac{1 - \ln x_n}{1 + x_n}\right]$$

n	1	2	3
x_n	0.5	0.5644	0.5671
$f(x_n)$	−0.1931	−0.0076	−0.0001

Approximate root: $x = 0.567$

87. $y = x\sqrt{x^2 - 1}$

$$\ln y = \ln x + \frac{1}{2}\ln(x^2 - 1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$\frac{dy}{dx} = y\left[\frac{2x^2 - 1}{x(x^2 - 1)}\right] = \frac{2x^2 - 1}{\sqrt{x^2 - 1}}$$

89. $$y = \frac{x^2\sqrt{3x-2}}{(x-1)^2}$$

$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x-1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x-1}$$

$$\frac{dy}{dx} = y\left[\frac{3x^2-15x+8}{2x(3x-2)(x-1)}\right]$$

$$= \frac{3x^3-15x^2+8x}{2(x-1)^3\sqrt{3x-2}}$$

91. $$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$

$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$

$$= \frac{y}{2}\left[\frac{4x^2+4x-2}{x(x^2-1)}\right] = \frac{(2x^2+2x-1)\sqrt{x-1}}{(x+1)^{3/2}}$$

93. Answers will vary. See Theorem 5.1 and 5.2.

95. $\ln e^x = x$ because $f(x) = \ln x$ and $g(x) = e^x$ are inverse functions.

97. (a) $f(1) \neq f(3)$

(b) $f'(x) = 1 - \frac{2}{x} = 0$ for $x = 2$.

99. $$\beta = 10\log_{10}\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}\ln\left(\frac{I}{10^{-16}}\right) = \frac{10}{\ln 10}[\ln I + 16\ln 10] = 160 + 10\log_{10} I$$

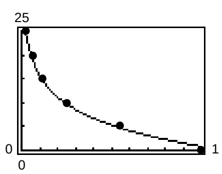
$$\beta(10^{-10}) = \frac{10}{\ln 10}[\ln 10^{-10} + 16\ln 10] = \frac{10}{\ln 10}[-10\ln 10 + 16\ln 10] = \frac{10}{\ln 10}[6\ln 10] = 60 \text{ decibels}$$

101. (a) You get an error message because $\ln h$ does not exist for $h = 0$.

(b) Reversing the data, you obtain

$h = 0.8627 - 6.4474 \ln p.$

(c)



(d) If $p = 0.75$, $h \approx 2.72$ km.

(e) If $h = 13$ km, $p \approx 0.15$ atmosphere.

(f) $h = 0.8627 - 6.4474 \ln p$

$$1 = -6.4474\frac{1}{p}\frac{dp}{dh} \quad \text{(implicit differentiation)}$$

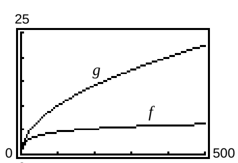
$$\frac{dp}{dh} = \frac{p}{-6.4474}$$

For $h = 5$, $p = 0.55$ and $dp/dh = -0.0853$ atmos/km.

For $h = 20$, $p = 0.06$ and $dp/dh = -0.00931$ atmos/km.

As the altitude increases, the rate of change of pressure decreases.

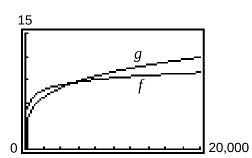
103. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$



$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{2\sqrt{x}}$$

For $x > 4$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x.

(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$



$$f'(x) = \frac{1}{x},\ g'(x) = \frac{1}{4\sqrt[4]{x^3}}$$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a faster rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.

105. False

$\ln x + \ln 25 = \ln(25x) \neq \ln(x + 25)$

Section 5.2 The Natural Logarithmic Function: Integration

1. $\int \frac{5}{x} dx = 5\int \frac{1}{x} dx = 5 \ln |x| + C$

3. $u = x + 1,\ du = dx$

$$\int \frac{1}{x+1}\, dx = \ln|x + 1| + C$$

5. $u = 3 - 2x,\ du = -2\, dx$

$$\int \frac{1}{3-2x}\, dx = -\frac{1}{2}\int \frac{1}{3-2x}(-2)\, dx$$
$$= -\frac{1}{2}\ln|3 - 2x| + C$$

7. $u = x^2 + 1,\ du = 2x\, dx$

$$\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{1}{x^2+1}(2x)\, dx$$
$$= \frac{1}{2}\ln(x^2 + 1) + C$$
$$= \ln\sqrt{x^2 + 1} + C$$

9. $\int \frac{x^2-4}{x}\, dx = \int \left(x - \frac{4}{x}\right) dx$

$$= \frac{x^2}{2} - 4\ln|x| + C$$

11. $u = x^3 + 3x^2 + 9x,\ du = 3(x^2 + 2x + 3)\, dx$

$$\int \frac{x^2+2x+3}{x^3+3x^2+9x}\, dx = \frac{1}{3}\int \frac{3(x^2+2x+3)}{x^3+3x^2+9x}\, dx$$
$$= \frac{1}{3}\ln|x^3 + 3x^2 + 9x| + C$$

13. $\int \frac{x^2-3x+2}{x+1} dx = \int \left(x - 4 + \frac{6}{x+1}\right) dx$

$$= \frac{x^2}{2} - 4x + 6\ln|x + 1| + C$$

15. $\int \frac{x^3-3x^2+5}{x-3}\, dx = \int \left(x^2 + \frac{5}{x-3}\right) dx$

$$= \frac{x^3}{3} + 5\ln|x - 3| + C$$

17. $\int \frac{x^4+x-4}{x^2+2} dx = \int \left(x^2 - 2 + \frac{x}{x^2+2}\right) dx$

$$= \frac{x^3}{3} - 2x + \frac{1}{2}\ln(x^2 + 2) + C$$

19. $u = \ln x,\ du = \frac{1}{x}\, dx$

$$\int \frac{(\ln x)^2}{x}\, dx = \frac{1}{3}(\ln x)^3 + C$$

21. $u = x + 1,\ du = dx$

$$\int \frac{1}{\sqrt{x+1}}\, dx = \int (x + 1)^{-1/2}\, dx$$
$$= 2(x + 1)^{1/2} + C$$
$$= 2\sqrt{x + 1} + C$$

23. $\int \frac{2x}{(x-1)^2}\, dx = \int \frac{2x-2+2}{(x-1)^2}\, dx$

$$= \int \frac{2(x-1)}{(x-1)^2}\, dx + 2\int \frac{1}{(x-1)^2}\, dx$$
$$= 2\int \frac{1}{x-1}\, dx + 2\int \frac{1}{(x-1)^2}\, dx$$
$$= 2\ln|x - 1| - \frac{2}{(x-1)} + C$$

25. $u = 1 + \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx \qquad (u - 1)\,du = dx$

$$\int \frac{1}{1 + \sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int \left(u - \frac{1}{u}\right) du$$
$$= u - \ln|u| + C_1$$
$$= \left(1 + \sqrt{2x}\right) - \ln\left|1 + \sqrt{2x}\right| + C_1$$
$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

27. $u = \sqrt{x} - 3,\ du = \dfrac{1}{2\sqrt{x}}\,dx \qquad 2(u + 3)\,du = dx$

$$\int \frac{\sqrt{x}}{\sqrt{x} - 3}\,dx = 2\int \frac{(u+3)^2}{u}\,du = 2\int \frac{u^2 + 6u + 9}{u}\,du = 2\int \left(u + 6 + \frac{9}{u}\right) du$$
$$= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 = u^2 + 12u + 18\ln|u| + C_1$$
$$= \left(\sqrt{x} - 3\right)^2 + 12\left(\sqrt{x} - 3\right) + 18\ln\left|\sqrt{x} - 3\right| + C_1$$
$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x} - 3\right| + C \text{ where } C = C_1 - 27.$$

29. $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,d\theta = \ln|\sin\theta| + C$

$(u = \sin\theta,\ du = \cos\theta\,d\theta)$

31. $\displaystyle\int \csc 2x\,dx = \frac{1}{2}\int (\csc 2x)(2)\,dx$

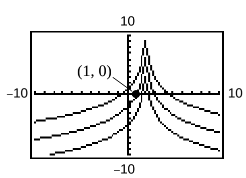
$$= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$$

33. $\displaystyle\int \frac{\cos t}{1 + \sin t}\,dt = \ln|1 + \sin t| + C$

35. $\displaystyle\int \frac{\sec x \tan x}{\sec x - 1}\,dx = \ln|\sec x - 1| + C$

37. $\displaystyle y = \int \frac{3}{2 - x}\,dx$

$$= -3\int \frac{1}{x - 2}\,dx$$
$$= -3\ln|x - 2| + C$$

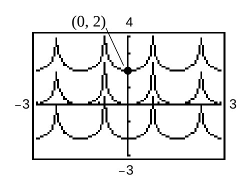


$(1, 0)$: $0 = -3\ln|1 - 2| + C \qquad C = 0$

$y = -3\ln|x - 2|$

39. $\displaystyle s = \int \tan(2\theta)\,d\theta$

$$= \frac{1}{2}\int \tan(2\theta)(2\,d\theta)$$
$$= -\frac{1}{2}\ln|\cos 2\theta| + C$$

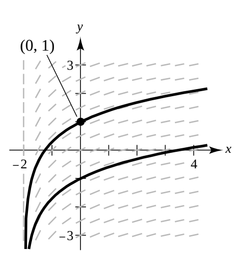


$(0, 2)$: $2 = -\frac{1}{2}\ln|\cos(0)| + C \qquad C = 2$

$s = -\frac{1}{2}\ln|\cos 2\theta| + 2$

41. $\dfrac{dy}{dx} = \dfrac{1}{x + 2},\ (0, 1)$

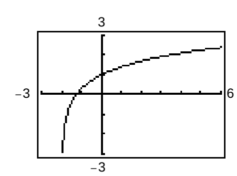
(a)



(b) $\displaystyle y = \int \frac{1}{x + 2}\,dx = \ln|x + 2| + C$

$y(0) = 1 \qquad 1 = \ln 2 + C \qquad C = 1 - \ln 2$

Hence, $y = \ln|x + 2| + 1 - \ln 2 = \ln\left|\dfrac{x + 2}{2}\right| + 1$.



43. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4$

$\displaystyle = \frac{5}{3}\ln 13 \approx 4.275$

45. $u = 1 + \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e = \frac{7}{3}$

47. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx = \int_0^2\left(x - 1 - \frac{1}{x+1}\right)dx$

$\displaystyle = \left[\frac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

49. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\,d\theta = \Big[\ln|\theta - \sin\theta|\Big]_1^2$

$\displaystyle = \ln\left|\frac{2-\sin 2}{1-\sin 1}\right| \approx 1.929$

51. $\displaystyle -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + \mathrm{C} = \ln|\sec x| + C$

53. $\displaystyle \ln|\sec x + \tan x| + C = \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C = \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C$

$\displaystyle = \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C$

55. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx = 2\left(1+\sqrt{x}\right) - 2\ln\left(1+\sqrt{x}\right) + C_1$

$= 2\left[\sqrt{x} - \ln\left(1+\sqrt{x}\right)\right] + C$ where $C = C_1 + 2$.

57. $\displaystyle\int \cos(1-x)\,dx = -\sin(1-x) + C$

59. $\displaystyle\int_{\pi/4}^{\pi/2}(\csc x - \sin x)\,dx = \Big[-\ln|\csc x + \cot x| + \cos x\Big]_{\pi/4}^{\pi/2} = \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2} \approx 0.174$

Note: In Exercises 61 and 63, you can use the Second Fundamental Theorem of Calculus or integrate the function.

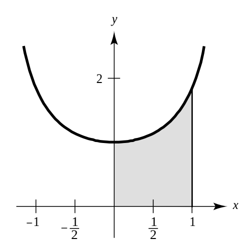
61. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

$\displaystyle F'(x) = \frac{1}{x}$

63. $\displaystyle F(x) = \int_x^{3x}\frac{1}{t}\,dt = \int_1^{3x}\frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt$

$\displaystyle F'(x) = \frac{3}{3x} - \frac{1}{x} = 0$

65.



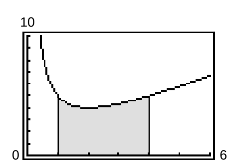
$A \approx 1.25$

Matches (d)

67. $\displaystyle A = \int_1^4 \frac{x^2+4}{x}\,dx = \int_1^4\left(x + \frac{4}{x}\right)dx$

$\displaystyle = \left[\frac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \frac{1}{2}$

$\displaystyle = \frac{15}{2} + 8\ln 2 \approx 13.045$ square units

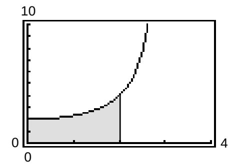


69. $\displaystyle\int_0^2 2\sec\frac{\pi x}{6}\,dx = \frac{12}{\pi}\int_0^2 \sec\left(\frac{\pi x}{6}\right)\frac{\pi}{6}\,dx$

$$= \left[\frac{12}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$$

$$= \frac{12}{\pi}\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{12}{\pi}\ln|1 + 0|$$

$$= \frac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.03041$$



71. Power Rule

73. Substitution: $(u = x^2 + 4)$ and Log Rule

75. Divide the polynomials:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

77. Average value $\displaystyle = \frac{1}{4-2}\int_2^4 \frac{8}{x^2}dx = 4\int_2^4 x^{-2}dx$

$$= \left[-4\frac{1}{x}\right]_2^4$$

$$= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1$$

79. Average value $\displaystyle = \frac{1}{e-1}\int_1^e \frac{\ln x}{x}dx = \frac{1}{e-1}\left[\frac{(\ln x)^2}{2}\right]_1^e$

$$= \frac{1}{e-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2e-2} \approx 0.291$$

81. $\displaystyle P(t) = \int \frac{3000}{1 + 0.25t}\,dt = (3000)(4)\int \frac{0.25}{1 + 0.25t}\,dt = 12{,}000\ln|1 + 0.25t| + C$

$P(0) = 12{,}000\ln|1 + 0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000\ln|1 + 0.25t| + 1000 = 1000[12\ln|1 + 0.25t| + 1]$

$P(3) = 1000[12(\ln 1.75) + 1] \approx 7715$

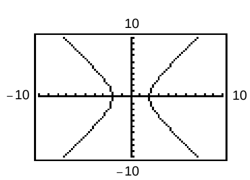
83. $\displaystyle \frac{1}{50 - 40}\int_{40}^{50} \frac{90{,}000}{400 + 3x}\,dx = \Big[3000\ln|400 + 3x|\Big]_{40}^{50} \approx \$168.27$

85. (a) $2x^2 - y^2 = 8$

$y^2 = 2x^2 - 8$

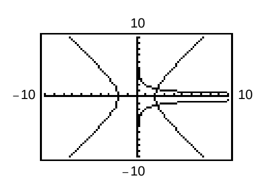
$y_1 = \sqrt{2x^2 - 8}$

$y_2 = -\sqrt{2x^2 - 8}$



(b) $\displaystyle y^2 = e^{-\int(1/x)dx} = e^{-\ln x + C} = e^{\ln(1/x)}(e^C) = \frac{1}{x}k$

Let $k = 4$ and graph $\displaystyle y^2 = \frac{4}{x}$ $\quad \begin{pmatrix} y_1 = 2/\sqrt{x} \\ y_2 = -2/\sqrt{x} \end{pmatrix}$



(c) In part (a), $2x^2 - y^2 = 8$

$4x - 2yy' = 0$

$\displaystyle y' = \frac{2x}{y}.$

In part (b), $\displaystyle y^2 = \frac{4}{x} = 4x^{-1}$

$\displaystyle 2yy' = \frac{-4}{x^2}$

$\displaystyle y' = \frac{-2}{yx^2} = \frac{-2y}{y^2x^2} = \frac{-2y}{4x} = \frac{-y}{2x}.$

Using a graphing utility the graphs intersect at $(2.214, 1.344)$. The slopes are 3.295 and $-0.304 = (-1)/3.295$, respectively.

87. False

$$\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$$

89. True

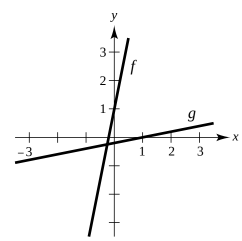
$$\int \frac{1}{x}\,dx = \ln|x| + C_1$$
$$= \ln|x| + \ln|C| = \ln|Cx|, \; C \neq 0$$

Section 5.3 Inverse Functions

1. (a)

$$f(x) = 5x + 1$$
$$g(x) = \frac{x-1}{5}$$
$$f(g(x)) = f\left(\frac{x-1}{5}\right) = 5\left(\frac{x-1}{5}\right) + 1 = x$$
$$g((f(x)) = g(5x+1) = \frac{(5x+1)-1}{5} = x$$

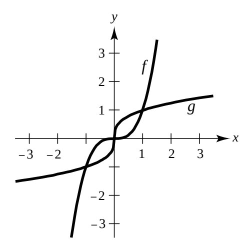
(b)



3. (a)

$$f(x) = x^3$$
$$g(x) = \sqrt[3]{x}$$
$$f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$$
$$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

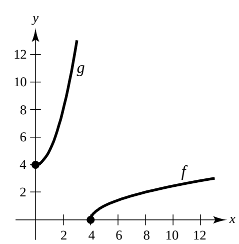
(b)



5. (a)

$$f(x) = \sqrt{x-4}$$
$$g(x) = x^2 + 4, \; x \quad 0$$
$$f(g(x)) = f(x^2 + 4)$$
$$= \sqrt{(x^2+4) - 4} = \sqrt{x^2} = x$$
$$g(f(x)) = g\left(\sqrt{x-4}\right)$$
$$= \left(\sqrt{x-4}\right)^2 + 4 = x - 4 + 4 = x$$

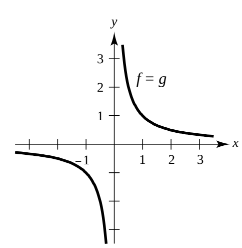
(b)



7. (a)

$$f(x) = \frac{1}{x}$$
$$g(x) = \frac{1}{x}$$
$$f(g(x)) = \frac{1}{1/x} = x$$
$$g(f(x)) = \frac{1}{1/x} = x$$

(b)

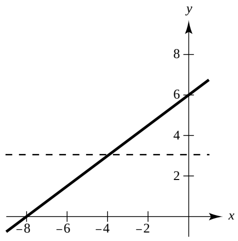


9. Matches (c)

11. Matches (a)

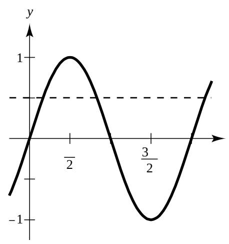
13. $f(x) = \frac{3}{4}x + 6$

One-to-one; has an inverse



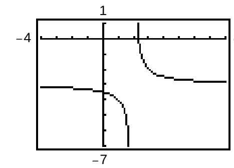
15. $f(\theta) = \sin\theta$

Not one-to-one; does not have an inverse



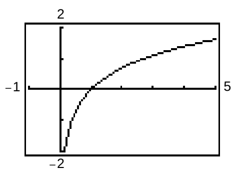
17. $h(s) = \dfrac{1}{s-2} - 3$

One-to-one; has an inverse



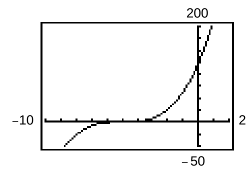
19. $f(x) = \ln x$

One-to-one; has an inverse



21. $g(x) = (x+5)^3$

One-to-one; has an inverse



23. $f(x) = (x+a)^3 + b$

$f'(x) = 3(x+a)^2 \quad 0$ for all x.

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

25. $f(x) = \dfrac{x^4}{4} - 2x^2$

$f'(x) = x^3 - 4x = 0$ when $x = 0, 2, -2$.

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

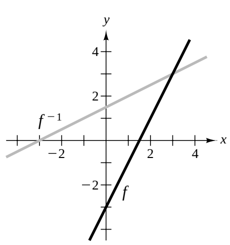
27. $f(x) = 2 - x - x^3$

$f'(x) = -1 - 3x^2 < 0$ for all x.

f is decreasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

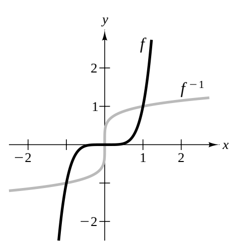
29.

$$f(x) = 2x - 3 = y$$
$$x = \frac{y+3}{2}$$
$$y = \frac{x+3}{2}$$
$$f^{-1}(x) = \frac{x+3}{2}$$



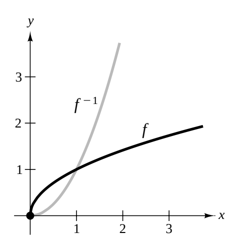
31.

$$f(x) = x^5 = y$$
$$x = \sqrt[5]{y}$$
$$y = \sqrt[5]{x}$$
$$f^{-1}(x) = \sqrt[5]{x} = x^{1/5}$$



33.

$$f(x) = \sqrt{x} = y$$
$$x = y^2$$
$$y = x^2$$
$$f^{-1}(x) = x^2, \quad x \quad 0$$

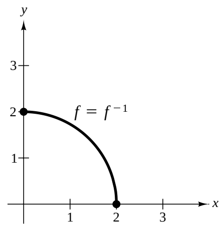


35. $f(x) = \sqrt{4 - x^2} = y, \; 0 \quad x \quad 2$

$x = \sqrt{4 - y^2}$

$y = \sqrt{4 - x^2}$

$f^{-1}(x) = \sqrt{4 - x^2}, \; 0 \quad x \quad 2$

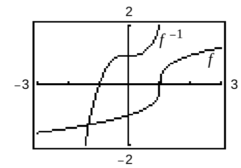


37. $f(x) = \sqrt[3]{x - 1} = y$

$x = y^3 + 1$

$y = x^3 + 1$

$f^{-1}(x) = x^3 + 1$



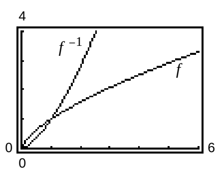
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

39. $f(x) = x^{2/3} = y, \; x \quad 0$

$x = y^{3/2}$

$y = x^{3/2}$

$f^{-1}(x) = x^{3/2}, \; x \quad 0$



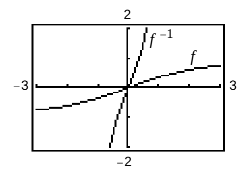
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

41. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}} = y$

$x = \dfrac{\sqrt{7}y}{\sqrt{1 - y^2}}$

$y = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}$

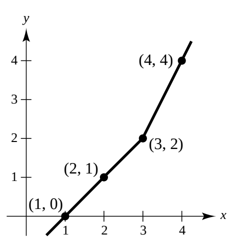
$f^{-1}(x) = \dfrac{\sqrt{7}x}{\sqrt{1 - x^2}}, \; -1 < x < 1$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

43.

x	1	2	3	4
$f^{-1}(x)$	0	1	2	4



45. (a) Let x be the number of pounds of the commodity costing 1.25 per pound. Since there are 50 pounds total, the amount of the second commodity is $50 - x$. The total cost is

$$y = 1.25x + 1.60(50 - x)$$
$$= -0.35x + 80 \qquad 0 \quad x \quad 50.$$

(b) We find the inverse of the original function:

$$y = -0.35x + 80$$
$$0.35x = 80 - y$$
$$x = \tfrac{100}{35}(80 - y)$$

Inverse: $y = \frac{100}{35}(80 - x) = \frac{20}{7}(80 - x)$.

x represents cost and y represents pounds.

(c) Domain of inverse is $62.5 \quad x \quad 80$.

(d) If $x = 73$ in the inverse function, $y = \frac{100}{35}(80 - 73) = \frac{100}{5} = 20$ pounds.

47. $f(x) = (x - 4)^2$ on $[4, \infty)$

$f'(x) = 2(x - 4) > 0$ on $(4, \infty)$

f is increasing on $[4, \infty)$. Therefore, f is strictly monotonic and has an inverse.

49. $f(x) = \dfrac{4}{x^2}$ on $(0, \infty)$

$f'(x) = -\dfrac{8}{x^3} < 0$ on $(0, \infty)$

f is decreasing on $(0, \infty)$. Therefore, f is strictly monotonic and has an inverse.

51. $f(x) = \cos x$ on $[0, \pi]$

$f'(x) = -\sin x < 0$ on $(0, \pi)$

f is decreasing on $[0, \pi]$. Therefore, f is strictly monotonic and has an inverse.

53. $f(x) = \dfrac{x}{x^2 - 4} = y$ on $(-2, 2)$

$$x^2y - 4y = x$$

$$x^2y - x - 4y = 0$$

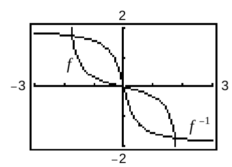
$$a = y, b = -1, c = -4y$$

$$x = \frac{1 \pm \sqrt{1 - 4(y)(-4y)}}{2y} = \frac{1 \pm \sqrt{1 + 16y^2}}{2y}$$

$$y = f^{-1}(x) = \begin{cases} \left(1 - \sqrt{1 + 16x^2}\right)/2x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

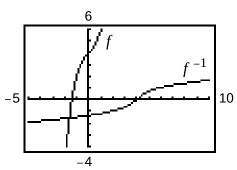
Domain: all x

Range: $-2 < y < 2$



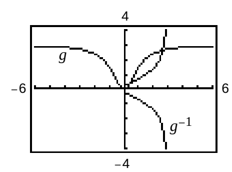
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

55. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

57. (a), (b)



(c) g is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

59. $f(x) = \sqrt{x - 2}$, Domain: $x \quad 2$

$f'(x) = \dfrac{1}{2\sqrt{x - 2}} > 0$ for $x > 2$.

f is one-to-one; has an inverse

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2$$

$$x = y^2 + 2$$

$$y = x^2 + 2$$

$$f^{-1}(x) = x^2 + 2, x \quad 0$$

61. $f(x) = |x - 2|, x \quad 2$

$$= -(x - 2)$$

$$= 2 - x$$

f is one-to-one; has an inverse

$$2 - x = y$$

$$2 - y = x$$

$$f^{-1}(x) = 2 - x, \ x \quad 0$$

63. $f(x) = (x - 3)^2$ is one-to-one for $x \quad 3$.

$$(x - 3)^2 = y$$

$$x - 3 = \sqrt{y}$$

$$x = \sqrt{y} + 3$$

$$y = \sqrt{x} + 3$$

$$f^{-1}(x) = \sqrt{x} + 3, \ x \quad 0$$

(Answer is not unique)

65. $f(x) = |x + 3|$ is one-to-one for $x \quad -3$.

$$x + 3 = y$$

$$x = y - 3$$

$$y = x - 3$$

$$f^{-1}(x) = x - 3, \ x \quad 0$$

(Answer is not unique)

67. Yes, the volume is an increasing function, and hence one-to-one. The inverse function gives the time t corresponding to the volume V.

69. No, $C(t)$ is not one-to-one because long distance costs are step functions. A call lasting 2.1 minutes costs the same as one lasting 2.2 minutes.

71.
$$f(x) = x^3 + 2x - 1, \; f(1) = 2 = a$$
$$f'(x) = 3x^2 + 2$$
$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

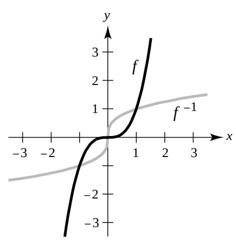
73.
$$f(x) = \sin x, f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$$
$$f'(x) = \cos x$$
$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)}$$
$$= \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

75.
$$f(x) = x^3 - \frac{4}{x}, \; f(2) = 6 = a$$
$$f'(x) = 3x^2 + \frac{4}{x^2}$$
$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2)^2 + (4/2^2)} = \frac{1}{13}$$

77. (a) Domain f = Domain $f^{-1} = (-\infty, \infty)$

(b) Range f = Range $f^{-1} = (-\infty, \infty)$

(c)

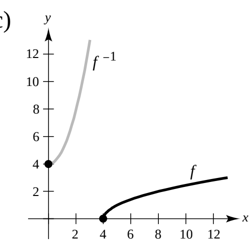


(d)
$$f(x) = x^3, \; \left(\frac{1}{2}, \frac{1}{8}\right)$$
$$f'(x) = 3x^2$$
$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$
$$f^{-1}(x) = \sqrt[3]{x}, \; \left(\frac{1}{8}, \frac{1}{2}\right)$$
$$(f^{-1})'(x) = \frac{1}{3\sqrt[3]{x^2}}$$
$$(f^{-1})'\left(\frac{1}{8}\right) = \frac{4}{3}$$

79. (a) Domain $f = [4, \infty)$, Domain $f^{-1} = [0, \infty)$

(b) Range $f = [0, \infty)$, Range $f^{-1} = [4, \infty)$

(c)



(d)
$$f(x) = \sqrt{x - 4}, \; (5, 1)$$
$$f'(x) = \frac{1}{2\sqrt{x - 4}}$$
$$f'(5) = \frac{1}{2}$$
$$f^{-1}(x) = x^2 + 4, \; (1, 5)$$
$$(f^{-1})'(x) = 2x$$
$$(f^{-1})'(1) = 2$$

81. $x = y^3 - 7y^2 + 2$

$1 = 3y^2 \dfrac{dy}{dx} - 14y \dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{3y^2 - 14y}$. At $(-4, 1)$, $\dfrac{dy}{dx} = \dfrac{1}{3 - 14} = \dfrac{-1}{11}$.

Alternate solution: let $f(x) = x^3 - 7x^2 + 2$.

Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$.

Hence, $\dfrac{dy}{dx} = \dfrac{1}{-11} = \dfrac{-1}{11}$.

In Exercises 83 and 85, use the following.

$\boldsymbol{f(x) = \frac{1}{8}x - 3}$ **and** $\boldsymbol{g(x) = x^3}$

$\boldsymbol{f^{-1}(x) = 8(x + 3)}$ **and** $\boldsymbol{g^{-1}(x) = \sqrt[3]{x}}$

83. $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

85. $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

In Exercises 87 and 89, use the following.

$\boldsymbol{f(x) = x + 4}$ **and** $\boldsymbol{g(x) = 2x - 5}$

$\boldsymbol{f^{-1}(x) = x - 4}$ **and** $\boldsymbol{g^{-1}(x) = \dfrac{x + 5}{2}}$

87. $(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x))$

$= g^{-1}(x - 4)$

$= \dfrac{(x - 4) + 5}{2}$

$= \dfrac{x + 1}{2}$

89. $(f \circ g)(x) = f(g(x))$

$= f(2x - 5)$

$= (2x - 5) + 4$

$= 2x - 1$

Hence, $(f \circ g)^{-1}(x) = \dfrac{x + 1}{2}$

(Note: $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)

91. Answers will vary. See page 335 and Example 3.

93. $y = x^2$ on $(-\infty, \infty)$ does not have an inverse.

95. f is not one-to-one because many different x-values yield the same y-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\dfrac{(2n - 1)\pi}{2}$, where n is an integer

97. Let $(f \circ g)(x) = y$ then $x = (f \circ g)^{-1}(y)$. Also,

$(f \circ g)(x) = y$

$f(g(x)) = y$

$g(x) = f^{-1}(y)$

$x = g^{-1}(f^{-1}(y))$

$= (g^{-1} \circ f^{-1})(y)$

Since f and g are one-to-one functions, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

99. Suppose $g(x)$ and $h(x)$ are both inverses of $f(x)$. Then the graph of $f(x)$ contains the point (a, b) if and only if the graphs of $g(x)$ and $h(x)$ contain the point (b, a). Since the graphs of $g(x)$ and $h(x)$ are the same, $g(x) = h(x)$. Therefore, the inverse of $f(x)$ is unique.

101. False

Let $f(x) = x^2$.

103. True

105. Not true

Let $f(x) = \begin{cases} x, & 0 \quad x \quad 1 \\ 1 - x, & 1 < x \quad 2 \end{cases}$.

f is one-to-one, but not strictly monotonic.

107.

$$f(x) = \int_2^x \frac{dt}{\sqrt{1 + t^4}},\ f(2) = 0$$

$$f'(x) = \frac{1}{\sqrt{1 + x^4}}$$

$$(f^{-1})'(0) = \frac{1}{f'(2)} = \frac{1}{1/\sqrt{17}} = \sqrt{17}$$

Section 5.4 Exponential Functions: Differentiation and Integration

1. $e^0 = 1$

$\ln 1 = 0$

3. $\ln 2 = 0.6931$

$e^{0.6931\ldots} = 2$

5. $e^{\ln x} = 4$

$x = 4$

7. $e^x = 12$

$x = \ln 12 \approx 2.485$

9. $9 - 2e^x = 7$

$2e^x = 7$

$e^x = 1$

$x = 0$

11. $50e^{-x} = 30$

$$e^{-x} = \frac{3}{5}$$

$$-x = \ln\left(\frac{3}{5}\right)$$

$$x = \ln\left(\frac{5}{3}\right) \approx 0.511$$

13. $\ln x = 2$

$x = e^2 \approx 7.3891$

15. $\ln(x - 3) = 2$

$x - 3 = e^2$

$x = 3 + e^2 \approx 10.389$

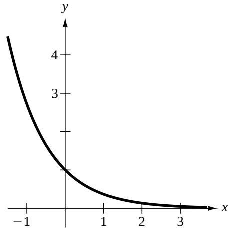
17. $\ln\sqrt{x + 2} = 1$

$\sqrt{x + 2} = e^1 = e$

$x + 2 = e^2$

$x = e^2 - 2 \approx 5.389$

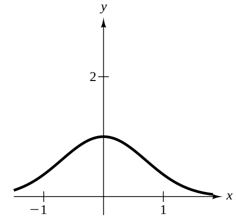
19. $y = e^{-x}$



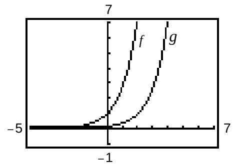
21. $y = e^{-x^2}$

Symmetric with respect to the y-axis

Horizontal asymptote: $y = 0$

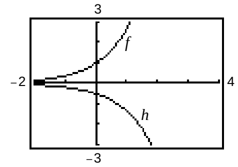


23. (a)



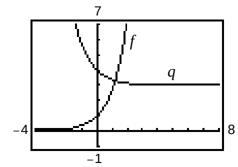
Horizontal shift 2 units to the right

(b)



A reflection in the x-axis and a vertical shrink

(c)



Vertical shift 3 units upward and a reflection in the y-axis

25. $y = Ce^{ax}$

Horizontal asymptote: $y = 0$

Matches (c)

27. $y = C(1 - e^{-ax})$

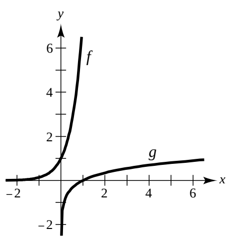
Vertical shift C units

Reflection in both the x- and y-axes

Matches (a)

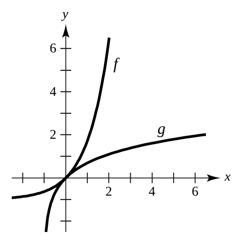
29. $f(x) = e^{2x}$

$g(x) = \ln \sqrt{x} = \frac{1}{2} \ln x$

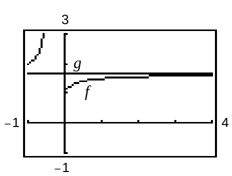


31. $f(x) = e^x - 1$

$g(x) = \ln(x + 1)$



33.



As $x \to \infty$, the graph of f approaches the graph of g.

$$\lim_{x \to \infty} \left(1 + \frac{0.5}{x}\right)^x = e^{0.5}$$

35. $\left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.718280469$

$e \approx 2.718281828$

$e > \left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

37. (a) $y = e^{3x}$

$y' = 3e^{3x}$

At $(0, 1)$, $y' = 3$.

(b) $y = e^{-3x}$

$y' = -3e^{-3x}$

At $(0, 1)$, $y' = -3$.

39. $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

41. $f(x) = e^{-2x + x^2}$

$\frac{dy}{dx} = 2(x - 1)e^{-2x + x^2}$

43. $y = e^{\sqrt{x}}$

$\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

45. $g(t) = (e^{-t} + e^t)^3$

$g'(t) = 3(e^{-t} + e^t)^2(e^t - e^{-t})$

47. $y = \ln e^{x^2} = x^2$

$\frac{dy}{dx} = 2x$

49. $y = \ln(1 + e^{2x})$

$\frac{dy}{dx} = \frac{2e^{2x}}{1 + e^{2x}}$

51. $y = \dfrac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$\dfrac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x})$

$= \dfrac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$

53. $y = x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2)$

$\dfrac{dy}{dx} = e^x(2x - 2) + e^x(x^2 - 2x + 2) = x^2e^x$

55. $f(x) = e^{-x}\ln x$

$f'(x) = e^{-x}\left(\dfrac{1}{x}\right) - e^{-x}\ln x = e^{-x}\left(\dfrac{1}{x} - \ln x\right)$

57. $y = e^x(\sin x + \cos x)$

$\dfrac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)(e^x)$

$= e^x(2\cos x) = 2e^x\cos x$

59. $xe^y - 10x + 3y = 0$

$xe^y\dfrac{dy}{dx} + e^y - 10 + 3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(xe^y + 3) = 10 - e^y$

$\dfrac{dy}{dx} = \dfrac{10 - e^y}{xe^y + 3}$

61. $f(x) = (3 + 2x)e^{-3x}$

$f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x}$

$= (-7 - 6x)e^{-3x}$

$f''(x) = (-7 - 6x)(-3e^{-3x}) - 6e^{-3x}$

$= 3(6x + 5)e^{-3x}$

63. $y = e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$

$y' = e^x\left(-\sqrt{2}\sin\sqrt{2}x + \sqrt{2}\cos\sqrt{2}x\right) + e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$

$= e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$

$y'' = e^x\left[-\left(\sqrt{2} + 2\right)\sin\sqrt{2}x + \left(\sqrt{2} - 2\right)\cos\sqrt{2}x\right] + e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right]$

$= e^x\left[\left(-1 - 2\sqrt{2}\right)\sin\sqrt{2}x + \left(-1 + 2\sqrt{2}\right)\cos\sqrt{2}x\right]$

$-2y' + 3y = -2e^x\left[\left(1 + \sqrt{2}\right)\cos\sqrt{2}x + \left(1 - \sqrt{2}\right)\sin\sqrt{2}x\right] + 3e^x\left[\cos\sqrt{2}x + \sin\sqrt{2}x\right]$

$= e^x\left[\left(1 - 2\sqrt{2}\right)\cos\sqrt{2}x + \left(1 + 2\sqrt{2}\right)\sin\sqrt{2}x\right] = -y''$

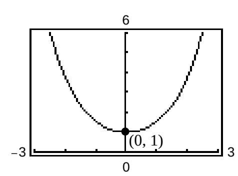
Therefore, $-2y' + 3y = -y''$ $\quad y'' - 2y' + 3y = 0.$

65. $f(x) = \dfrac{e^x + e^{-x}}{2}$

$f'(x) = \dfrac{e^x - e^{-x}}{2} = 0$ when $x = 0$.

$f''(x) = \dfrac{e^x + e^{-x}}{2} > 0$

Relative minimum: $(0, 1)$



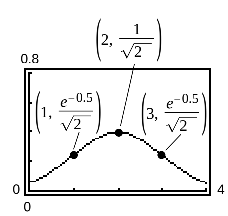
67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$

$g'(x) = \dfrac{-1}{\sqrt{2\pi}}(x - 2)e^{-(x-2)^2/2}$

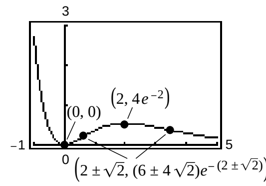
$g''(x) = \dfrac{1}{\sqrt{2\pi}}(x - 1)(x - 3)e^{-(x-2)^2/2}$

Relative maximum: $\left(2, \dfrac{1}{\sqrt{2\pi}}\right) \approx (2, 0.399)$

Points of inflection: $\left(1, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(3, \dfrac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (1, 0.242), (3, 0.242)$



69. $f(x) = x^2e^{-x}$



$f'(x) = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2 - x) = 0$ when $x = 0, 2$.

$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$

$= e^{-x}(x^2 - 4x + 2) = 0$ when $x = 2 \pm \sqrt{2}$.

Relative minimum: $(0, 0)$

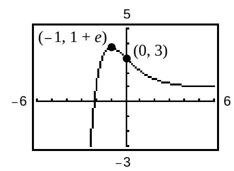
Relative maximum: $(2, 4e^{-2})$

$x = 2 \pm \sqrt{2}$

$y = (2 \pm \sqrt{2})^2e^{-(2 \pm \sqrt{2})}$

Points of inflection: $(3.414, 0.384), (0.586, 0.191)$

71. $g(t) = 1 + (2 + t)e^{-t}$



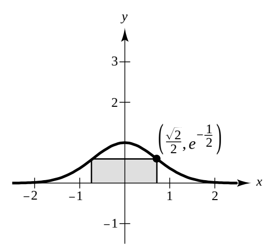
$g'(t) = (1 + t)e^{-t}$

$g''(t) = te^{-t}$

Relative maximum: $(-1, 1 + e) \approx (-1, 3.718)$

Point of inflection: $(0, 3)$

73. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$



$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}(1 - 2x^2) = 0 \text{ when } x = \frac{\sqrt{2}}{2}.$$

$A = \sqrt{2}e^{-1/2}$

75. $y = \dfrac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^2} = \frac{\frac{aL}{b}e^{-x/b}}{(1 + ae^{-x/b})^2}$$

$$y'' = \frac{(1 + ae^{-x/b})^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2(1 + ae^{-x/b})\left(\frac{-a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^4}$$

$$= \frac{(1 + ae^{-x/b})\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{(1 + ae^{-x/b})^3}$$

$$= \frac{Lae^{-x/b}[ae^{-x/b} - 1]}{(1 + ae^{-x/b})^3 b^2}$$

$y'' = 0$ if $ae^{-x/b} = 1 \qquad \dfrac{-x}{b} = \ln\left(\dfrac{1}{a}\right) \qquad x = b \ln a$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

Therefore, the y-coordinate of the inflection point is $L/2$.

77. $e^{-x} = x \qquad f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$x_1 = 1$

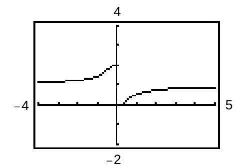
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.5379$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.5670$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 0.5671$$

We approximate the root of f to be $x = 0.567$.

79. (a)

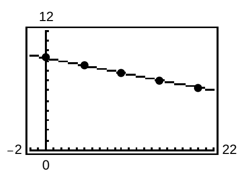


(b) When x increases without bound, $1/x$ approaches zero, and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1 + 1) = 1$. Thus, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞ and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist since the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

81.

h	0	5	10	15	20
P	10,332	5,583	2,376	1,240	517
$\ln P$	9.243	8.627	7.773	7.123	6.248

(a)



$y = -0.1499h + 9.3018$ is the regression line for data $(h, \ln P)$.

(b) $\ln P = ah + b$

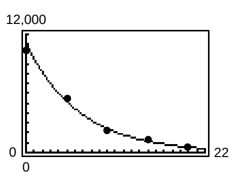
$P = e^{ah+b} = e^b e^{ah}$

$P = Ce^{ah}, C = e^b$

For our data, $a = -0.1499$ and $C = e^{9.3018} = 10{,}957.7$

$P = 10{,}957.7e^{-0.1499h}$

(c)



(d) $\dfrac{dP}{dh} = (10{,}957.71)(-0.1499)e^{-0.1499h}$

$= -1642.56e^{-0.1499h}$

For $h = 5$, $\dfrac{dP}{dh} = -776.3$. For $h = 18$, $\dfrac{dP}{dh} \approx -110.6$.

83. $f(x) = e^{x/2}, f(0) = 1$

$f'(x) = \frac{1}{2}e^{x/2}, f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{4}e^{x/2}, f''(0) = \frac{1}{4}$

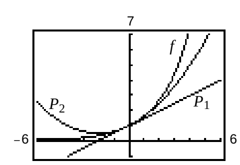
$P_1(x) = 1 + \frac{1}{2}(x - 0) = \frac{x}{2} + 1, P_1(0) = 1$

$P_1'(x) = \frac{1}{2}, P_1'(0) = \frac{1}{2}$

$P_2(x) = 1 + \frac{1}{2}(x - 0) + \frac{1}{8}(x - 0)^2 = \frac{x^2}{8} + \frac{x}{2} + 1, P_2(0) = 1$

$P_2'(x) = \frac{1}{4}x + \frac{1}{2}, P_2'(0) = \frac{1}{2}$

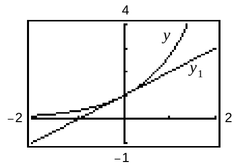
$P_2''(x) = \frac{1}{4}, P_2''(0) = \frac{1}{4}$



The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.

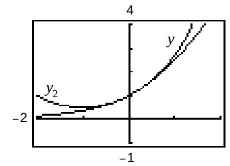
85. (a) $y = e^x$

$y_1 = 1 + x$



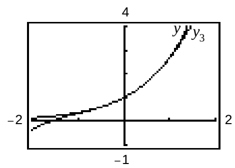
(b) $y = e^x$

$y_2 = 1 + x + \left(\frac{x^2}{2}\right)$



(c) $y = e^x$

$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$



87. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,5dx = e^{5x} + C$$

89. Let $u = -2x$, $du = -2\,dx$.

$$\int_0^1 e^{-2x}dx = -\frac{1}{2}\int_0^1 e^{-2x}(-2)\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$
$$= \frac{1}{2}(1 - e^{-2}) = \frac{e^2 - 1}{2e^2}$$

91. $\displaystyle\int xe^{-x^2}dx = -\frac{1}{2}\int e^{-x^2}(-2x)dx = -\frac{1}{2}e^{-x^2} + C$

93. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2e^{\sqrt{x}} + C$

95. Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$.

$$\int \frac{e^{-x}}{1 + e^{-x}}\,dx = -\int \frac{-e^{-x}}{1 + e^{-x}}\,dx = -\ln(1 + e^{-x}) + C = \ln\left(\frac{e^x}{e^x + 1}\right) + C = x - \ln(e^x + 1) + C$$

97. Let $u = \frac{3}{x}$, $du = -\frac{3}{x^2}\,dx$.

$$\int_1^3 \frac{e^{3/x}}{x^2}\,dx = -\frac{1}{3}\int_1^3 e^{3/x}\left(-\frac{3}{x^2}\right)dx$$
$$= \left[-\frac{1}{3}e^{3/x}\right]_1^3 = \frac{e}{3}(e^2 - 1)$$

99. Let $u = 1 - e^x$, $du = -e^x\,dx$.

$$\int e^x\sqrt{1 - e^x}\,dx = -\int (1 - e^x)^{1/2}(-e^x)\,dx$$
$$= -\frac{2}{3}(1 - e^x)^{3/2} + C$$

101. Let $u = e^x - e^{-x}$, $du = (e^x + e^{-x})\,dx$.

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \ln\left|e^x - e^{-x}\right| + C$$

103. $\displaystyle\int \frac{5 - e^x}{e^{2x}}\,dx = \int 5e^{-2x}\,dx - \int e^{-x}\,dx$

$$= -\frac{5}{2}e^{-2x} + e^{-x} + C$$

105. $\displaystyle\int e^{\sin \pi x}\cos \pi x\,dx = \frac{1}{\pi}\int e^{\sin \pi x}(\pi \cos \pi x)\,dx$

$$= \frac{1}{\pi}e^{\sin \pi x} + C$$

107. $\displaystyle\int e^{-x}\tan(e^{-x})\,dx = -\int [\tan(e^{-x})](-e^{-x})\,dx$

$$= \ln\left|\cos(e^{-x})\right| + C$$

109. Let $u = ax^2$, $du = 2ax\,dx$. (Assume $a \neq 0$)

$$y = \int xe^{ax^2}\,dx$$

$$= \frac{1}{2a}\int e^{ax^2}(2ax)\,dx = \frac{1}{2a}e^{ax^2} + C$$

111. $f'(x) = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x}) + C_1$

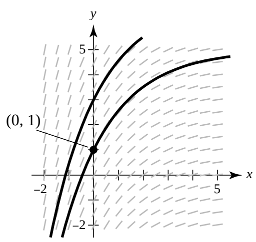
$f'(0) = C_1 = 0$

$f(x) = \int \frac{1}{2}(e^x - e^{-x})\,dx = \frac{1}{2}(e^x + e^{-x}) + C_2$

$f(0) = 1 + C_2 = 1 \qquad C_2 = 0$

$f(x) = \frac{1}{2}(e^x + e^{-x})$

113. (a)



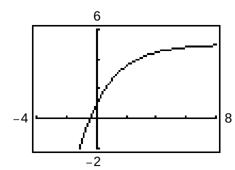
(b) $\frac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$

$$y = \int 2e^{-x/2}\,dx = -4\int e^{-x/2}\left(-\frac{1}{2}\,dx\right)$$

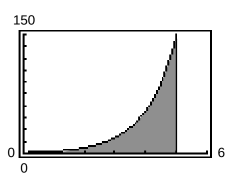
$$= -4e^{-x/2} + C$$

$(0, 1)$: $1 = -4e^0 + C = -4 + C \qquad C = 5$

$y = -4e^{-x/2} + 5$

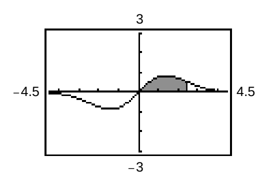


115. $\int_0^5 e^x\,dx = \left[e^x\right]_0^5 = e^5 - 1 \approx 147.413$



117. $\int_0^{\sqrt{6}} xe^{-x^2/4}dx = \left[-2e^{-x^2/4}\right]_0^{\sqrt{6}}$

$$= -2e^{-3/2} + 2 \approx 1.554$$



119. (a) $f(u - v) = e^{u-v} = (e^u)(e^{-v}) = \frac{e^u}{e^v} = \frac{f(u)}{f(v)}$

(b) $f(kx) = e^{kx} = (e^x)^k = [f(x)]^k$.

121. $0.0665\int_{48}^{60} e^{-0.0139(t-48)^2}\,dt$

Graphing Utility: $0.4772 = 47.72\%$

123. $\int_0^x e^t\,dt \qquad \int_0^x 1\,dt$

$\left[e^t\right]_0^x \qquad \left[t\right]_0^x$

$e^x - 1 \qquad x \qquad e^x \qquad 1 + x$ for $x \qquad 0$

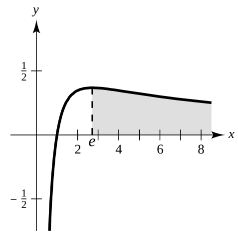
125. $f(x) = e^x$. Domain is $(-\infty, \infty)$ and range is $(0, \infty)$. f is continuous, increasing, one-to-one, and concave upwards on its entire domain.

$\lim_{x \quad -\infty} e^x = 0$ and $\lim_{x \quad \infty} e^x = \infty$.

127. Yes. $f(x) = Ce^x$, C a constant.

129. $e^{-x} > 0 \qquad \int_0^2 e^{-x}dx > 0$.

131. $f(x) = \dfrac{\ln x}{x}$



(a) $f'(x) = \dfrac{1 - \ln x}{x^2} = 0$ when $x = e$.

On $(0, e)$, $f'(x) > 0$ $\quad$ f is increasing.

On (e, ∞), $f'(x) < 0$ $\quad$ f is decreasing.

(b) For $e \quad A < B$, we have:

$$\frac{\ln A}{A} > \frac{\ln B}{B}$$

$$B \ln A > A \ln B$$

$$\ln A^B > \ln B^A$$

$$A^B > B^A.$$

(c) Since $e < \pi$, from part (b) we have $e^\pi > \pi^e$.

Section 5.5 Bases Other than *e* and Applications

1. $y = \left(\frac{1}{2}\right)^{t/3}$

At $t_0 = 6$, $y = \left(\frac{1}{2}\right)^{6/3} = \frac{1}{4}$

3. $y = \left(\frac{1}{2}\right)^{t/7}$

At $t_0 = 10$, $y = \left(\frac{1}{2}\right)^{10/7} \approx 0.3715$

5. $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$

7. $\log_7 1 = 0$

9. (a) $2^3 = 8$

$\log_2 8 = 3$

(b) $3^{-1} = \dfrac{1}{3}$

$\log_3 \dfrac{1}{3} = -1$

11. (a) $\log_{10} 0.01 = -2$

$10^{-2} = 0.01$

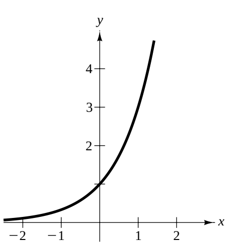
(b) $\log_{0.5} 8 = -3$

$0.5^{-3} = 8$

$\left(\frac{1}{2}\right)^{-3} = 8$

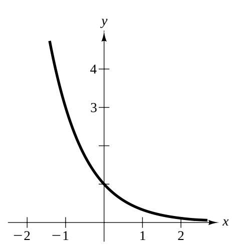
13. $y = 3^x$

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



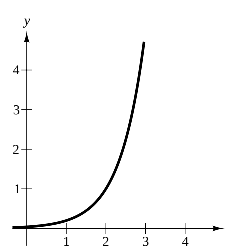
15. $y = \left(\frac{1}{3}\right)^x = 3^{-x}$

x	-2	-1	0	1	2
y	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



17. $h(x) = 5^{x-2}$

x	-1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5



19. (a) $\log_{10} 1000 = x$

$$10^x = 1000$$
$$x = 3$$

(b) $\log_{10} 0.1 = x$

$$10^x = 0.1$$
$$x = -1$$

21. (a) $\log_3 x = -1$

$$3^{-1} = x$$
$$x = \tfrac{1}{3}$$

(b) $\log_2 x = -4$

$$2^{-4} = x$$
$$x = \tfrac{1}{16}$$

23. (a)

$$x^2 - x = \log_5 25$$
$$x^2 - x = \log_5 5^2 = 2$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1 \text{ OR } x = 2$$

(b) $3x + 5 = \log_2 64$

$$3x + 5 = \log_2 2^6 = 6$$
$$3x = 1$$
$$x = \tfrac{1}{3}$$

25.

$$3^{2x} = 75$$
$$2x \ln 3 = \ln 75$$
$$x = \frac{1}{2}\frac{\ln 75}{\ln 3} \approx 1.965$$

27.

$$2^{3-x} = 625$$
$$(3 - x)\ln 2 = \ln 625$$
$$3 - x = \frac{\ln 625}{\ln 2}$$
$$x = 3 - \frac{\ln 625}{\ln 2} \approx -6.288$$

29.

$$\left(1 + \frac{0.09}{12}\right)^{12t} = 3$$
$$12t \ln\left(1 + \frac{0.09}{12}\right) = \ln 3$$
$$t = \frac{1}{12}\frac{\ln 3}{\ln\left(1 + \frac{0.09}{12}\right)} \approx 12.253$$

31. $\log_2(x - 1) = 5$

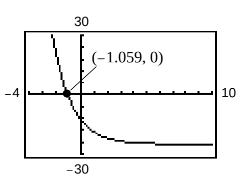
$$x - 1 = 2^5 = 32$$
$$x = 33$$

33. $\log_3 x^2 = 4.5$

$$x^2 = 3^{4.5}$$
$$x = \pm\sqrt{3^{4.5}} \approx \pm 11.845$$

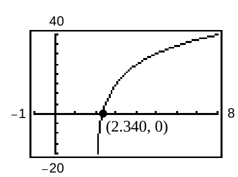
35. $g(x) = 6(2^{1-x}) - 25$

Zero: $x \approx -1.059$



37. $h(s) = 32 \log_{10}(s - 2) + 15$

Zero: $s \approx 2.340$

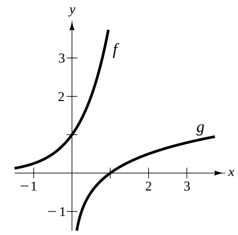


39. $f(x) = 4^x$

$g(x) = \log_4 x$

x	-2	-1	0	$\frac{1}{2}$	1
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4

x	$\frac{1}{16}$	$\frac{1}{4}$	1	2	4
$g(x)$	-2	-1	0	$\frac{1}{2}$	1



41. $f(x) = 4^x$

$f'(x) = (\ln 4)\, 4^x$

43. $y = 5^{x-2}$

$$\frac{dy}{dx} = (\ln 5)\, 5^{x-2}$$

45. $g(t) = t^2\, 2^t$

$$g'(t) = t^2 (\ln 2)\, 2^t + (2t)\, 2^t$$
$$= t\, 2^t (t \ln 2 + 2)$$
$$= 2^t t(2 + t \ln 2)$$

47. $h(\theta) = 2^{-\theta} \cos \pi\theta$

$$h'(\theta) = 2^{-\theta}(-\pi \sin \pi\theta) - (\ln 2)2^{-\theta} \cos \pi\theta$$
$$= -2^{-\theta}[(\ln 2) \cos \pi\theta + \pi \sin \pi\theta]$$

49. $y = \log_3 x$

$$\frac{dy}{dx} = \frac{1}{x \ln 3}$$

51. $f(x) = \log_2 \dfrac{x^2}{x-1}$

$$= 2 \log_2 x - \log_2 (x - 1)$$
$$f'(x) = \frac{2}{x \ln 2} - \frac{1}{(x-1) \ln 2}$$
$$= \frac{x-2}{(\ln 2)x(x-1)}$$

53. $y = \log_5 \sqrt{x^2 - 1} = \dfrac{1}{2} \log_5 (x^2 - 1)$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{2x}{(x^2-1)\ln 5} = \frac{x}{(x^2-1)\ln 5}$$

55. $g(t) = \dfrac{10 \log_4 t}{t} = \dfrac{10}{\ln 4}\left(\dfrac{\ln t}{t}\right)$

$$g'(t) = \frac{10}{\ln 4}\left[\frac{t(1/t) - \ln t}{t^2}\right]$$
$$= \frac{10}{t^2 \ln 4}[1 - \ln t] = \frac{5}{t^2 \ln 2}(1 - \ln t)$$

57. $y = x^{2/x}$

$$\ln y = \frac{2}{x} \ln x$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)$$
$$\frac{dy}{dx} = \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$$

59. $y = (x - 2)^{x+1}$

$$\ln y = (x + 1) \ln(x - 2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x+1)\left(\frac{1}{x-2}\right) + \ln(x - 2)$$
$$\frac{dy}{dx} = y\left[\frac{x+1}{x-2} + \ln(x - 2)\right]$$
$$= (x - 2)^{x+1}\left[\frac{x+1}{x-2} + \ln(x - 2)\right]$$

61. $\displaystyle\int 3^x\, dx = \frac{3^x}{\ln 3} + C$

63. $\displaystyle\int_{-1}^{2} 2^x\,dx = \left[\frac{2^x}{\ln 2}\right]_{-1}^{2}$

$\displaystyle = \frac{1}{\ln 2}\left[4 - \frac{1}{2}\right]$

$\displaystyle = \frac{7}{2\ln 2} = \frac{7}{\ln 4}$

65. $\displaystyle\int x\,5^{-x^2}\,dx = -\frac{1}{2}\int 5^{-x^2}(-2x)\,dx$

$\displaystyle = -\left(\frac{1}{2}\right)\frac{5^{-x^2}}{\ln 5} + C$

$\displaystyle = \frac{-1}{2\ln 5}(5^{-x^2}) + C$

67. $\displaystyle\int \frac{3^{2x}}{1 + 3^{2x}}\,dx,\ u = 1 + 3^{2x},\ du = 2(\ln 3)3^{2x}\,dx$

$\displaystyle \frac{1}{2\ln 3}\int \frac{(2\ln 3)3^{2x}}{1 + 3^{2x}}\,dx = \frac{1}{2\ln 3}\ln(1 + 3^{2x}) + C$

69. $\displaystyle\frac{dy}{dx} = 0.4^{x/3},\ \left(0, \frac{1}{2}\right)$

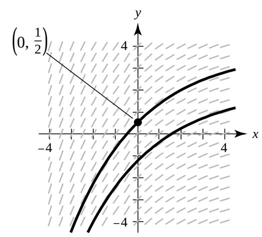
$\displaystyle y = \int 0.4^{x/3}\,dx = 3\int 0.4^{x/3}\left(\frac{1}{3}\,dx\right)$

$\displaystyle = \frac{3}{\ln 0.4}0.4^{x/3} + C = 3(\ln 2.5)(0.4)^{x/3} + C$

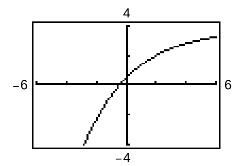
$\displaystyle y = 3\ln 2.5(0.4)^{x/3} + \frac{1}{2} - 3\ln 2.5$

$\displaystyle = \frac{3(1 - 0.4^{x/3})}{\ln 2.5} + \frac{1}{2}$

(a)

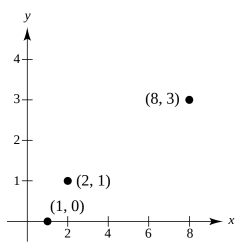


(b)



71. Answers will vary. Example: Growth and decay problems.

73.



x	1	2	8
y	0	1	3

(a) y is an exponential function of x: False

(b) y is a logarithmic function of x: True; $y = \log_2 x$

(c) x is an exponential function of y: True, $2^y = x$

(d) y is a linear function of x: False

75. $f(x) = \log_2 x \qquad f'(x) = \dfrac{1}{x\ln 2}$

$g(x) = x^x \qquad g'(x) = x^x(1 + \ln x)$

[**Note:** Let $y = g(x)$. Then: $\ln y = \ln x^x = x\ln x$

$\displaystyle \frac{1}{y}y' = x\cdot\frac{1}{x} + \ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x) = g'(x)$.]

$h(x) = x^2 \qquad h'(x) = 2x$

$k(x) = 2^x \qquad k'(x) = (\ln 2)2^x$

From greatest to smallest rate of growth:
$g(x), k(x), h(x), f(x)$

77. $C(t) = P(1.05)^t$

(a) $C(10) = 24.95(1.05)^{10}$

$\approx \$40.64$

(b) $\dfrac{dC}{dt} = P(\ln 1.05)(1.05)^t$

When $t = 1$: $\dfrac{dC}{dt} \approx 0.051P$

When $t = 8$: $\dfrac{dC}{dt} \approx 0.072P$

(c) $\dfrac{dC}{dt} = (\ln 1.05)[P(1.05)^t]$

$= (\ln 1.05)C(t)$

The constant of proportionality is $\ln 1.05$.

79. $P = \$1000,\ r = 3\frac{1}{2}\% = 0.035,\ t = 10$

$A = 1000\left(1 + \frac{0.035}{n}\right)^{10n}$

$A = 1000e^{(0.035)(10)} = 1419.07$

n	1	2	4	12	365	Continuous
A	1410.60	1414.78	1416.91	1418.34	1419.04	1419.07

81. $P = \$1000,\ r = 5\% = 0.05,\ t = 30$

$A = 1000\left(1 + \frac{0.05}{n}\right)^{30n}$

$A = 1000e^{(0.05)30} = 4481.69$

n	1	2	4	12	365	Continuous
A	4321.94	4399.79	4440.21	4467.74	4481.23	4481.69

83. $100{,}000 = Pe^{0.05t}$ $\quad P = 100{,}000e^{-0.05t}$

t	1	10	20	30	40	50
P	95,122.94	60,653.07	36,787.94	22,313.02	13,583.53	8208.50

85. $100{,}000 = P\left(1 + \frac{0.05}{12}\right)^{12t}$ $\quad P = 100{,}000\left(1 + \frac{0.05}{12}\right)^{-12t}$

t	1	10	20	30	40	50
P	95,132.82	60,716.10	36,864.45	22,382.66	13,589.88	8251.24

87. (a) $A = 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} \approx \$32{,}320.21$

(b) $A = \$30{,}000$

(c) $A = 8000\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + 20{,}000\left(1 + \frac{0.06}{365}\right)^{(365)(4)}$

$\approx \$12{,}928.09 + 25{,}424.48 = \$38{,}352.57$

(d) $A = 9000\left[\left(1 + \frac{0.06}{365}\right)^{(365)(8)} + \left(1 + \frac{0.06}{365}\right)^{(365)(4)} + 1\right]$

$\approx \$34{,}985.11$

Take option (c).

89. (a) $\lim_{t \to \infty} 6.7e^{(-48.1)/t} = 6.7e^0 = 6.7$ million ft^3

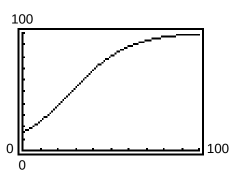
(b) $V' = \frac{322.27}{t^2} e^{-(48.1)/t}$

$V'(20) \approx 0.073$ million ft^3/yr

$V'(60) \approx 0.040$ million ft^3/yr

91. $y = \frac{300}{3 + 17e^{-0.0625x}}$

(a)



(b) If $x = 2$ (2000 egg masses), $y \approx 16.67 \approx 16.7\%$.

(c) If $y = 66.67\%$, then $x \approx 38.8$ or 38,800 egg masses.

(d) $y = 300(3 + 17e^{-0.0625x})^{-1}$

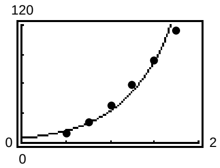
$y' = \frac{318.75e^{-0.0625x}}{(3 + 17e^{-0.0625x})^2}$

$y'' = \frac{19.921875e^{-0.0625x}(17e^{-0.0625x} - 3)}{(3 + 17e^{-0.0625x})^3}$

$17e^{-0.0625x} - 3 = 0$ $\quad x \approx 27.8$ or 27,800 egg masses.

93. (a) $B = 4.7539(6.7744)^d = 4.7539e^{1.9132d}$

(b)



(c) $B'(d) = 9.0952e^{1.9132d}$

$B'(0.8) \approx 42.03$ tons/inch

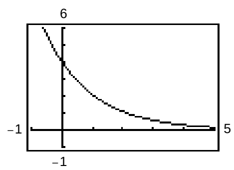
$B'(1.5) \approx 160.38$ tons/inch

95. (a) $\int_0^4 f(t)\,dt \approx 5.67$

$\int_0^4 g(t)\,dt \approx 5.67$

$\int_0^4 h(t)\,dt \approx 5.67$

(b)



(c) The functions appear to be equal: $f(t) = g(t) = h(t)$

Analytically,

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3} = 4\left[\left(\frac{3}{8}\right)^{2/3}\right]^t = 4\left(\frac{9^{1/3}}{4}\right)^t = g(t)$$

$$h(t) = 4e^{-0.653886t} = 4[e^{-0.653886}]^t = 4(0.52002)^t$$

$$g(t) = 4\left(\frac{9^{1/3}}{4}\right)^t = 4(0.52002)^t$$

No. The definite integrals over a given interval may be equal when the functions are not equal.

97. $$P = \int_0^{10} 2000e^{-0.06t}\,dt$$

$$= \left[\frac{2000}{-0.06}e^{-0.06t}\right]_0^{10}$$

$$\approx \$15{,}039.61$$

99.

t	0	1	2	3	4
y	1200	720	432	259.20	155.52

$y = C(k^t)$

When $t = 0, y = 1200$ $\quad C = 1200.$

$y = 1200(k^t)$

$$\frac{720}{1200} = 0.6,\ \frac{432}{720} = 0.6,\ \frac{259.20}{432} = 0.6,\ \frac{155.52}{259.20} = 0.6$$

Let $k = 0.6$.

$y = 1200(0.6)^t$

101. False. e is an irrational number.

103. True.

$$f(g(x)) = 2 + e^{\ln(x-2)}$$
$$= 2 + x - 2 = x$$
$$g(f(x)) = \ln(2 + e^x - 2)$$
$$= \ln e^x = x$$

105. True.

$\frac{d}{dx}[e^x] = e^x$ and $\frac{d}{dx}[e^{-x}] = -e^{-x}$

$e^x = e^{-x}$ when $x = 0$.

$(e^0)(-e^{-0}) = -1$

107. $$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), y(0) = 1$$

$$\frac{dy}{y[(5/4) - y]} = \frac{8}{25}\,dt \qquad \frac{4}{5}\int\left(\frac{1}{y} + \frac{1}{(5/4) - y}\right)dy = \int\frac{8}{25}\,dt$$

$$\ln y - \ln\left(\frac{5}{4} - y\right) = \frac{2}{5}t + C$$

$$\ln\left(\frac{y}{(5/4) - y}\right) = \frac{2}{5}t + C$$

$$\frac{y}{(5/4) - y} = e^{(2/5)t + C} = C_1e^{(2/5)t}$$

$$y(0) = 1 \qquad C_1 = 4 \qquad 4e^{(2/5)t} = \frac{y}{(5/4) - y}$$

$$4e^{(2/5)t}\left(\frac{5}{4} - y\right) = y \qquad 5e^{(2/5)t} = 4e^{(2/5)t}y + y = (4e^{(2/5)t} + 1)y$$

$$y = \frac{5e^{(2/5)t}}{4e^{(2/5)t} + 1} = \frac{5}{4 + e^{-0.4t}} = \frac{1.25}{1 + 0.25e^{-0.4t}}$$

Section 5.6 Differential Equations: Growth and Decay

1. $$\frac{dy}{dx} = x + 2$$

$$y = \int(x + 2)dx = \frac{x^2}{2} + 2x + C$$

3. $$\frac{dy}{dx} = y + 2$$

$$\frac{dy}{y + 2} = dx$$

$$\int\frac{1}{y + 2}dy = \int dx$$

$$\ln|y + 2| = x + C_1$$

$$y + 2 = e^{x + C_1} = Ce^x$$

$$y = Ce^x - 2$$

5. $$y' = \frac{5x}{y}$$

$$yy' = 5x$$

$$\int yy'\,dx = \int 5x\,dx$$

$$\int y\,dy = \int 5x\,dx$$

$$\frac{1}{2}y^2 = \frac{5}{2}x^2 + C_1$$

$$y^2 - 5x^2 = C$$

7. $$y' = \sqrt{x}\,y$$

$$\frac{y'}{y} = \sqrt{x}$$

$$\int\frac{y'}{y}\,dx = \int\sqrt{x}\,dx$$

$$\int\frac{dy}{y} = \int\sqrt{x}\,dx$$

$$\ln y = \frac{2}{3}x^{3/2} + C_1$$

$$y = e^{(2/3)x^{3/2} + C_1}$$

$$= e^{C_1}e^{(2/3)x^{3/2}}$$

$$= Ce^{(2/3)x^{3/2}}$$

9. $(1 + x^2)y' - 2xy = 0$

$$y' = \frac{2xy}{1 + x^2}$$
$$\frac{y'}{y} = \frac{2x}{1 + x^2}$$
$$\int \frac{y'}{y}\,dx = \int \frac{2x}{1 + x^2}\,dx$$
$$\int \frac{dy}{y} = \int \frac{2x}{1 + x^2}\,dx$$
$$\ln y = \ln(1 + x^2) + C_1$$
$$\ln y = \ln(1 + x^2) + \ln C$$
$$\ln y = \ln C(1 + x^2)$$
$$y = C(1 + x^2)$$

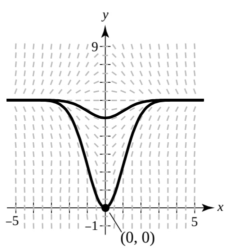
11. $\dfrac{dQ}{dt} = \dfrac{k}{t^2}$

$$\int \frac{dQ}{dt}\,dt = \int \frac{k}{t^2}\,dt$$
$$\int dQ = -\frac{k}{t} + C$$
$$Q = -\frac{k}{t} + C$$

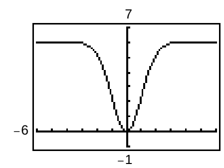
13. $\dfrac{dN}{ds} = k(250 - s)$

$$\int \frac{dN}{ds}\,ds = \int k(250 - s)\,ds$$
$$\int dN = -\frac{k}{2}(250 - s)^2 + C$$
$$N = -\frac{k}{2}(250 - s)^2 + C$$

15. (a)



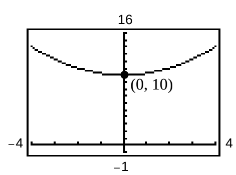
(b) $\dfrac{dy}{dx} = x(6 - y)$, $(0, 0)$



$$\frac{dy}{y - 6} = -x$$
$$\ln|y - 6| = \frac{-x^2}{2} + C$$
$$y - 6 = e^{-x^2/2 + C} = C_1e^{-x^2/2}$$
$$y = 6 + C_1e^{-x^2/2}$$

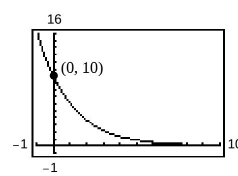
$(0, 0)$: $0 = 6 + C_1$ $\quad C_1 = -6 \quad y = 6 - 6e^{-x^2/2}$

17. $\dfrac{dy}{dt} = \dfrac{1}{2}t$, $(0, 10)$



$$\int dy = \int \frac{1}{2}t\,dt$$
$$y = \frac{1}{4}t^2 + C$$
$$10 = \frac{1}{4}(0)^2 + C \qquad C = 10$$
$$y = \frac{1}{4}t^2 + 10$$

19. $\dfrac{dy}{dt} = -\dfrac{1}{2}y$, $(0, 10)$



$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dt$$
$$\ln y = -\frac{1}{2}t + C_1$$
$$y = e^{-(t/2) + C_1} = e^{C_1}\,e^{-t/2} = Ce^{-t/2}$$
$$10 = Ce^0 \qquad C = 10$$
$$y = 10e^{-t/2}$$

21. $\dfrac{dy}{dx} = ky$

$y = Ce^{kx}$ (Theorem 5.16)

$(0, 4)$: $4 = Ce^0 = C$

$(3, 10)$: $10 = 4e^{3k} \qquad k = \dfrac{1}{3}\ln\left(\dfrac{5}{2}\right)$

When $x = 6$, $y = 4e^{1/3\ln(5/2)(6)} = 4e^{\ln(5/2)^2}$

$$= 4\left(\frac{5}{2}\right)^2 = 25$$

23. $\dfrac{dV}{dt} = kV$

$V = Ce^{kt}$ (Theorem 5.16)

$(0, 20{,}000)$: $C = 20{,}000$

$(4, 12{,}500)$: $12{,}500 = 20{,}000e^{4k} \qquad k = \dfrac{1}{4}\ln\left(\dfrac{5}{8}\right)$

When $t = 6$, $V = 20{,}000e^{1/4\ln(5/8)(6)} = 20{,}000e^{\ln(5/8)^{3/2}}$

$$= 20{,}000\left(\frac{5}{8}\right)^{3/2} \approx 9882.118$$

25. $y = Ce^{kt}, \left(0, \frac{1}{2}\right), (5, 5)$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{kt}$$

$$5 = \frac{1}{2}e^{5k}$$

$$k = \frac{\ln 10}{5} \approx 0.4605$$

$$y = \frac{1}{2}e^{0.4605t}$$

27. $y = Ce^{kt},\ (1, 1),\ (5, 5)$

$$1 = Ce^{k}$$

$$5 = Ce^{5k}$$

$$5Ce^{k} = Ce^{5k}$$

$$5e^{k} = e^{5k}$$

$$5 = e^{4k}$$

$$k = \frac{\ln 5}{4} \approx 0.4024$$

$$y = Ce^{0.4024t}$$

$$1 = Ce^{0.4024}$$

$$C \approx 0.6687$$

$$y = 0.6687e^{0.4024t}$$

29. A differential equation in x and y is an equation that involves x, y and derivatives of y.

31. $\frac{dy}{dx} = \frac{1}{2}xy$

$\frac{dy}{dx} > 0$ when $xy > 0$. Quadrants I and III.

33. Since the initial quantity is 10 grams, $y = 10e^{[\ln(1/2)/1620]t}$. When $t = 1000$, $y = 10e^{[\ln(1/2)/1620](1000)} \approx 6.52$ grams. When $t = 10{,}000$, $y = 10e^{[\ln(1/2)/1620](10{,}000)} \approx 0.14$ gram.

35. Since $y = Ce^{[\ln(1/2)/1620]t}$, we have $0.5 = Ce^{[\ln(1/2)/1620](10{,}000)}$ $C \approx 36.07$.

Initial quantity: 36.07 grams.

When $t = 1000$, we have $y = Ce^{[\ln(1/2)/1620](1000)} \approx 23.51$ grams.

37. Since the initial quantity is 5 grams, we have $y = 5.0e^{[\ln(1/2)/5730]t}$.

When $t = 1000$, $y \approx 4.43$ g.

When $t = 10{,}000$, $y \approx 1.49$ g.

39. Since $y = Ce^{[\ln(1/2)/24{,}360]t}$, we have $2.1 = Ce^{[\ln(1/2)/24{,}360](1000)}$ $C \approx 2.16$. Thus, the initial quantity is 2.16 grams. When $t = 10{,}000$, $y = 2.16e^{[\ln(1/2)/24{,}360](10{,}000)} \approx 1.63$ grams.

41. Since $\frac{dy}{dx} = ky,\ y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{1620k}$$

$$k = \frac{-\ln 2}{1620}$$

$$y = y_0e^{-(\ln 2)t/1620}.$$

When $t = 100$, $y = y_0e^{-(\ln 2)/16.2} \approx y_0(0.9581)$. Therefore, 95.81% of the present amount still exists.

43. Since $A = 1000e^{0.06t}$, the time to double is given by $2000 = 1000e^{0.06t}$ and we have

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years.}$$

Amount after 10 years: $A = 1000e^{(0.06)(10)} \approx \$1822.12$

45. Since $A = 750e^{rt}$ and $A = 1500$ when $t = 7.75$, we have the following.

$$1500 = 750e^{7.75r}$$

$$r = \frac{\ln 2}{7.75} \approx 0.0894 = 8.94\%$$

Amount after 10 years: $A = 750e^{0.0894(10)} \approx \$1833.67$

47. Since $A = 500e^{rt}$ and $A = 1292.85$ when $t = 10$, we have the following.

$$1292.85 = 500e^{10r}$$

$$r = \frac{\ln(1292.85/500)}{10} \approx 0.0950 = 9.50\%$$

The time to double is given by

$$1000 = 500e^{0.0950t}$$

$$t = \frac{\ln 2}{0.095} \approx 7.30 \text{ years.}$$

49. $500{,}000 = P\left(1 + \frac{0.075}{12}\right)^{(12)(20)}$

$$P = 500{,}000\left(1 + \frac{0.075}{12}\right)^{-240}$$

$$\approx \$112{,}087.09$$

51. $500{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(35)}$

$$P = 500{,}000\left(1 + \frac{0.08}{12}\right)^{-420}$$

$$= \$30{,}688.87$$

53. (a) $2000 = 1000(1 + 0.07)^t$

$$2 = 1.07^t$$

$$\ln 2 = t \ln 1.07$$

$$t = \frac{\ln 2}{\ln 1.07} \approx 10.24 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.07}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.007}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.07}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln(1 + (0.07/12))} \approx 9.93 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.07}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.07}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.07}{365}\right)$$

$$t = \frac{\ln 2}{365 \ln(1 + (0.07/365))} \approx 9.90 \text{ years}$$

(d) $2000 = 1000e^{(0.07)t}$

$$2 = e^{0.07t}$$

$$\ln 2 = 0.07t$$

$$t = \frac{\ln 2}{0.07} \approx 9.90 \text{ years}$$

55. (a) $2000 = 1000(1 + 0.085)^t$

$$2 = 1.085^t$$

$$\ln 2 = t \ln 1.085$$

$$t = \frac{\ln 2}{\ln 1.085} \approx 8.50 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.085}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.085}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.085}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{12}\right)} \approx 8.18 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.085}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.085}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.085}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.085}{365}\right)} \approx 8.16 \text{ years}$$

(d) $2000 = 1000e^{0.085t}$

$$2 = e^{0.085t}$$

$$\ln 2 = 0.085t$$

$$t = \frac{\ln 2}{0.085} \approx 8.15 \text{ years}$$

57. $P = Ce^{kt} = Ce^{-0.009t}$

$P(-1) = 8.2 = Ce^{-0.009(-1)} \qquad C = 8.1265$

$P = 8.1265e^{-0.009t}$

$P(10) \approx 7.43$ or 7,430,000 people in 2010

59. $P = Ce^{kt} = Ce^{0.036t}$

$P(-1) = 4.6 = Ce^{0.036(-1)} \qquad C = 4.7686$

$P = 4.7686e^{0.036t}$

$P(10) \approx 6.83$ or 6,830,000 people in 2010

61. If $k < 0$, the population decreases.

If $k > 0$, the population increases.

63.
$$P = Ce^{kx}, \ (0, 760), \ (1000, 672.71)$$
$$C = 760$$
$$672.71 = 760e^{1000x}$$
$$x = \frac{\ln(672.71/760)}{1000} \approx -0.000122$$
$$P \approx 760e^{-0.000122x}$$

When $x = 3000$, $P \approx 527.06$ mm Hg.

65. (a)
$$19 = 30(1 - e^{20k})$$
$$30e^{20k} = 11$$
$$k = \frac{\ln(11/30)}{20} \approx -0.0502$$
$$N \approx 30(1 - e^{-0.0502t})$$

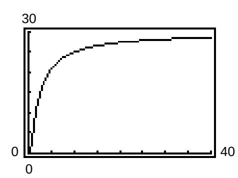
(b)
$$25 = 30(1 - e^{-0.0502t})$$
$$e^{-0.0502t} = \frac{1}{6}$$
$$t = \frac{-\ln 6}{-0.0502} \approx 36 \text{ days}$$

67. $S = Ce^{k/t}$

(a)
$$S = 5 \text{ when } t = 1$$
$$5 = Ce^{k}$$
$$\lim_{t \ \infty} Ce^{k/t} = C = 30$$
$$5 = 30e^{k}$$
$$k = \ln \tfrac{1}{6} \approx -1.7918$$
$$S \approx 30e^{-1.7918/t}$$

(b) When $t = 5$, $S \approx 20.9646$ which is 20,965 units.

(c)



69. $A(t) = V(t)e^{-0.10t} = 100{,}000e^{0.8\sqrt{t}}\, e^{-0.10t} = 100{,}000e^{0.8\sqrt{t} - 0.10t}$

$$\frac{dA}{dt} = 100{,}000\left(\frac{0.4}{\sqrt{t}} - 0.10\right)e^{0.8\sqrt{t} - 0.10t} = 0 \text{ when } 16.$$

The timber should be harvested in the year 2014, $(1998 + 16)$. **Note:** You could also use a graphing utility to graph $A(t)$ and find the maximum of $A(t)$. Use the viewing rectangle $0 \quad x \quad 30$ and $0 \quad y \quad 600{,}000$.

71. $\beta(I) = 10 \log_{10} \dfrac{I}{I_0}, I_0 = 10^{-16}$

(a) $\beta(10^{-14}) = 10 \log_{10} \dfrac{10^{-14}}{10^{-16}} = 20$ decibels

(b) $\beta(10^{-9}) = 10 \log_{10} \dfrac{10^{-9}}{10^{-16}} = 70$ decibels

(c) $\beta(10^{-6.5}) = 10 \log_{10} \dfrac{10^{-6.5}}{10^{-16}} = 95$ decibels

(d) $\beta(10^{-4}) = 10 \log_{10} \dfrac{10^{-4}}{10^{-16}} = 120$ decibels

73. $R = \dfrac{\ln I - 0}{\ln 10}, I = e^{R \ln 10} = 10^{R}$

(a) $8.3 = \dfrac{\ln I - 0}{\ln 10}$

$$I = 10^{8.3} \approx 199{,}526{,}231.5$$

(b) $2R = \dfrac{\ln I - 0}{\ln 10}$

$$I = e^{2R \ln 10} = e^{2R \ln 10} = (e^{R \ln 10})^2 = (10^{R})^2$$

Increases by a factor of $e^{2R \ln 10}$ or 10^{R}.

(c) $\dfrac{dR}{dI} = \dfrac{1}{I \ln 10}$

75. False. If $y = Ce^{kt}$, $y' = Cke^{kt} \neq$ constant.

77. True

Section 5.7 Differential Equations: Separation of Variables

1. Differential equation: $y' = 4y$

Solution: $y = Ce^{4x}$

Check: $y' = 4Ce^{4x} = 4y$

3. Differential equation: $y'' + y = 0$

Solution: $y = C_1 \cos x + C_2 \sin x$

Check:
$$y' = -C_1 \sin x + C_2 \cos x$$
$$y'' = -C_1 \cos x - C_2 \sin x$$
$$y'' + y = -C_1 \cos x - C_2 \sin x + C_1 \cos x + C_2 \sin x = 0$$

5. $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$
$$= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$
$$= -1 + \sin x \ln|\sec x + \tan x|$$
$$y'' = (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$
$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Substituting,

$$y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x|$$
$$= \tan x.$$

In Exercises 7–11, the differential equation is $y^{(4)} - 16y = 0$.

7.
$$y = 3 \cos x$$
$$y^{(4)} = 3 \cos x$$
$$y^{(4)} - 16y = -45 \cos x \neq 0,$$

No.

9.
$$y = e^{-2x}$$
$$y^{(4)} = 16e^{-2x}$$
$$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$$

Yes.

11.
$$y = C_1e^{2x} + C_2e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$$
$$y^{(4)} = 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3 \sin 2x + 16C_4 \cos 2x$$
$$y^{(4)} - 16y = 0,$$

Yes.

In 13–17, the differential equation is $xy' - 2y = x^3e^x$.

13. $y = x^2,\ y' = 2x$

$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x$

No.

15. $y = x^2(2 + e^x),\ \ y' = x^2(e^x) + 2x(2 + e^x)$

$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2] = x^3e^x,$

Yes.

17. $y = \ln x, y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2 \ln x \neq x^3e^x,$ No.

19. $y = Ce^{kx}$

$\dfrac{dy}{dx} = Cke^{kx}$

Since $dy/dx = 0.07y$, we have $Cke^{kx} = 0.07Ce^{kx}$.
Thus, $k = 0.07$.

21. $y^2 = Cx^3$ passes through $(4, 4)$

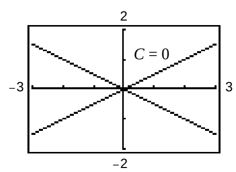
$16 = C(64) \qquad C = \frac{1}{4}$

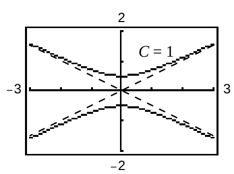
Particular solution: $y^2 = \frac{1}{4}x^3$ or $4y^2 = x^3$

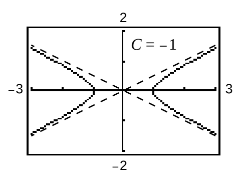
23. Differential equation: $4yy' - x = 0$

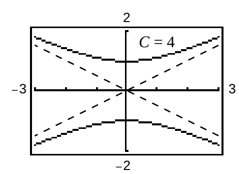
General solution: $4y^2 - x^2 = C$

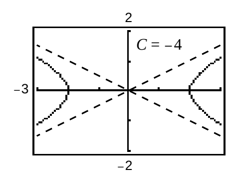
Particular solutions: $C = 0$, Two intersecting lines
$C = \pm 1,\ C = \pm 4$, Hyperbolas











25. Differential equation: $y' + 2y = 0$

General Solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition: $y(0) = 3, 3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

27. Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$

$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$

$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) +$
$9(C_1 \sin 3x + C_2 \cos 3x) = 0$

Initial conditions: $y\left(\dfrac{\pi}{6}\right) = 2,\ y'\left(\dfrac{\pi}{6}\right) = 1$

$2 = C_1 \sin\left(\dfrac{\pi}{2}\right) + C_2 \cos\left(\dfrac{\pi}{2}\right) \qquad C_1 = 2$

$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$

$1 = 3C_1 \cos\left(\dfrac{\pi}{2}\right) - 3C_2 \sin\left(\dfrac{\pi}{2}\right)$

$= -3C_2 \qquad C_2 = -\dfrac{1}{3}$

Particular solution: $y = 2 \sin 3x - \dfrac{1}{3} \cos 3x$

29. Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + 3(C_1x + C_2x^3) = 0$$

Initial conditions: $y(2) = 0,\ y'(2) = 4$

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$$\left.\begin{aligned} C_1 + 4C_2 &= 0 \\ C_1 + 12C_2 &= 4 \end{aligned}\right\} \quad C_2 = \frac{1}{2},\quad C_1 = -2$$

Particular solution: $y = -2x + \frac{1}{2}x^3$

31. $\frac{dy}{dx} = 3x^2$

$$y = \int 3x^2\,dx = x^3 + C$$

33. $\frac{dy}{dx} = \frac{x}{1 + x^2}$

$$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln(1 + x^2) + C$$

$(u = 1 + x^2, du = 2x\,dx)$

35. $\frac{dy}{dx} = \frac{x - 2}{x} = 1 - \frac{2}{x}$

$$y = \int\left[1 - \frac{2}{x}\right]dx$$

$$= x - 2\ln|x| + C = x - \ln x^2 + C$$

37. $\frac{dy}{dx} = \sin 2x$

$$y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$(u = 2x, du = 2dx)$

39. $\frac{dy}{dx} = x\sqrt{x - 3}$ Let $u = \sqrt{x - 3}$, then $x = u^2 + 3$ and $dx = 2u\,du$.

$$y = \int x\sqrt{x - 3}\,dx = \int (u^2 + 3)(u)(2u)du$$

$$= 2\int (u^4 + 3u^2)\,du = 2\left(\frac{u^5}{5} + u^3\right) + C = \frac{2}{5}(x - 3)^{5/2} + 2(x - 3)^{3/2} + C$$

41. $\frac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

$(u = x^2,\ du = 2x\,dx)$

43. $\frac{dy}{dx} = \frac{x}{y}$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = C$$

45. $\frac{dr}{ds} = 0.05r$

$$\int \frac{dr}{r} = \int 0.05\,ds$$

$$\ln|r| = 0.05s + C_1$$

$$r = e^{0.05s + C_1} = Ce^{0.05s}$$

47. $(2 + x)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$$

$$\ln y = 3\ln(2 + x) + \ln C = \ln C(2 + x)^3$$

$$y = C(x + 2)^3$$

49. $yy' = \sin x$

$$\int y\,dy = \int \sin x\,dx$$

$$\frac{y^2}{2} = -\cos x + C_1$$

$$y^2 = -2\cos x + C$$

51. $\sqrt{1-4x^2}\,\dfrac{dy}{dx} = x$

$$dy = \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$\int dy = \int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$$= -\frac{1}{8}\int (1-4x^2)^{-1/2}(-8x\,dx)$$

$$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$$

53. $y\ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln y = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

55. $yy' - e^x = 0$

$$\int y\,dy = \int e^x\,dx$$

$$\frac{y^2}{2} = e^x + C_1$$

$$y^2 = 2e^x + C$$

Initial condition: $y(0) = 4,\ 16 = 2 + C,\ C = 14$

Particular solution: $y^2 = 2e^x + 14$

57. $y(x+1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x+1)\,dx$$

$$\ln y = -\frac{(x+1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition: $y(-2) = 1,\ 1 = Ce^{-1/2},\ C = e^{1/2}$

Particular solution: $y = e^{[1-(x+1)^2]/2} = e^{-(x^2+2x)/2}$

59. $y(1+x^2)\dfrac{dy}{dx} = x(1+y^2)$

$$\frac{y}{1+y^2}\,dy = \frac{x}{1+x^2}\,dx$$

$$\frac{1}{2}\ln(1+y^2) = \frac{1}{2}\ln(1+x^2) + C_1$$

$$\ln(1+y^2) = \ln(1+x^2) + \ln C = \ln[C(1+x^2)]$$

$$1 + y^2 = C(1+x^2)$$

$$y(0) = \sqrt{3}:\ 1 + 3 = C \qquad C = 4$$

$$1 + y^2 = 4(1+x^2)$$

$$y^2 = 3 + 4x^2$$

61. $\dfrac{du}{dv} = uv\sin v^2$

$$\int \frac{du}{u} = \int v\sin v^2\,dv$$

$$\ln u = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-(\cos v^2)/2}$$

Initial condition: $u(0) = 1,\ C = \dfrac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{(1-\cos v^2)/2}$

63. $dP - kP\,dt = 0$

$$\int \frac{dP}{P} = k\int dt$$

$$\ln P = kt + C_1$$

$$P = Ce^{kt}$$

Initial condition: $P(0) = P_0,\ P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

65. $$\frac{dy}{dx} = \frac{-9x}{16y}$$

$$\int 16y\,dy = -\int 9x\,dx$$

$$8y^2 = \frac{-9}{2}x^2 + C$$

Initial condition: $y(1) = 1,\ 8 = -\frac{9}{2} + C,\ C = \frac{25}{2}$

Particular solution: $8y^2 = \frac{-9}{2}x^2 + \frac{25}{2},$

$$16y^2 + 9x^2 = 25$$

67. $$m = \frac{dy}{dx} = \frac{0 - y}{(x + 2) - x} = -\frac{y}{2}$$

$$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$$

$$\ln y = -\frac{1}{2}x + C_1$$

$$y = Ce^{-x/2}$$

69. $f(x, y) = x^3 - 4xy^2 + y^3$

$$f(tx, ty) = t^3x^3 - 4txt^2y^2 + t^3y^3$$
$$= t^3(x^3 - 4xy^2 + y^3)$$

Homogeneous of degree 3

71. $f(x, y) = \frac{x^2y^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{t^4x^2y^2}{\sqrt{t^2x^2 + t^2y^2}} = t^3\frac{x^2y^2}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 3

73. $f(x, y) = 2\ln xy$

$$f(tx, ty) = 2\ln tx\,ty$$
$$= 2\ln t^2xy = 2(\ln t^2 + \ln xy)$$

Not homogeneous

75. $f(x, y) = 2\ln\frac{x}{y}$

$$f(tx, ty) = 2\ln\frac{tx}{ty} = 2\ln\frac{x}{y}$$

Homogeneous degree 0

77. $$y' = \frac{x + y}{2x},\ y = vx$$

$$v + x\frac{dv}{dx} = \frac{x + vx}{2x}$$

$$x\frac{dv}{dx} = \frac{1 + v}{2} - v$$

$$2\int\frac{dv}{1 - v} = \int\frac{dx}{x}$$

$$-\ln(1 - v)^2 = \ln|x| + \ln C = \ln|Cx|$$

$$\frac{1}{(1 - v^2)} = |Cx|$$

$$\frac{1}{[1 - (y/x)]^2} = |Cx|$$

$$\frac{x^2}{(x - y)^2} = |Cx|$$

$$|x| = C(x - y)^2$$

79. $$y' = \frac{x - y}{x + y},\ y = vx$$

$$v + x\frac{dv}{dx} = \frac{x - xv}{x + xv}$$

$$v\,dx + x\,dv = \frac{1 - v}{1 + v}dx$$

$$\int\frac{v + 1}{v^2 + 2v - 1}\,dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\ln|v^2 + 2v - 1| = -\ln|x| + \ln C_1 = \ln\left|\frac{C_1}{x}\right|$$

$$|v^2 + 2v - 1| = \frac{C}{x^2}$$

$$\left|\frac{y^2}{x^2} + 2\frac{y}{x} - 1\right| = \frac{C}{x^2}$$

$$|y^2 + 2xy - x^2| = C$$

81. $$y' = \frac{xy}{x^2 - y^2}, y = vx$$

$$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2 - x^2 v^2}$$

$$v\,dx + x\,dv = \frac{v}{1 - v^2}\,dx$$

$$\int \frac{1 - v^2}{v^3}\,dv = \int \frac{dx}{x}$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + \ln C_1 = \ln|C_1 x|$$

$$\frac{-1}{2v^2} = \ln|C_1 x v|$$

$$\frac{-x^2}{2y^2} = \ln|C_1 y|$$

$$y = Ce^{-x^2/2y^2}$$

83. $$x\,dy - (2xe^{-y/x} + y)\,dx = 0, y = vx$$

$$x(v\,dx + x\,dv) - (2xe^{-v} + vx)\,dx = 0$$

$$\int e^v\,dv = \int \frac{2}{x}\,dx$$

$$e^v = \ln C_1 x^2$$

$$e^{y/x} = \ln C_1 + \ln x^2$$

$$e^{y/x} = C + \ln x^2$$

Initial condition: $y(1) = 0, 1 = C$

Particular solution: $e^{y/x} = 1 + \ln x^2$

85. $$\left(x \sec\frac{y}{x} + y\right)dx - x\,dy = 0, y = vx$$

$$(x \sec v + xv)dx - x(v\,dx + x\,dv) = 0$$

$$(\sec v + v)\,dx = v\,dx + x\,dv$$

$$\int \cos v\,dv = \int \frac{dx}{x}$$

$$\sin v = \ln x + \ln C_1$$

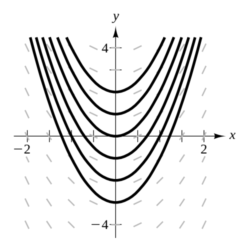
$$x = Ce^{\sin v}$$

$$= Ce^{\sin(y/x)}$$

Initial condition: $y(1) = 0, 1 = Ce^0 = C$

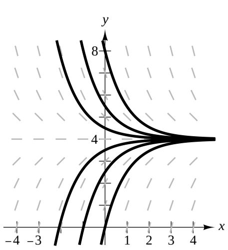
Particular solution: $x = e^{\sin(y/x)}$

87. $$\frac{dy}{dx} = x$$



$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

89. $$\frac{dy}{dx} = 4 - y$$



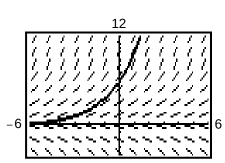
$$\int \frac{dy}{4 - y} = \int dx$$

$$\ln|4 - y| = -x + C_1$$

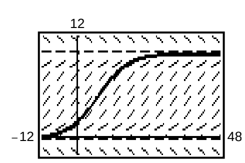
$$4 - y = e^{-x + C_1}$$

$$y = 4 + Ce^{-x}$$

91. $$\frac{dy}{dx} = 0.5y, y(0) = 6$$



93. $$\frac{dy}{dx} = 0.02y(10 - y), y(0) = 2$$



95. $\dfrac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = y_0$

$$y(1620) = \frac{y_0}{2}$$

$$C = y_0$$

$$\frac{y_0}{2} = y_0 e^{1620k}$$

$$k = \frac{\ln(1/2)}{1620}$$

Particular solution: $y = y_0 e^{-t(\ln 2)/1620}$

When $t = 25$, $y \approx 0.989y_0$, $y = 98.9\%$ of y_0.

97. $\dfrac{dy}{dx} = k(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

99. $\dfrac{dy}{dx} = ky(y - 4)$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

101.
$$\frac{dw}{dt} = k(1200 - w)$$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$

$$\ln(1200 - w) = -kt + C_1$$

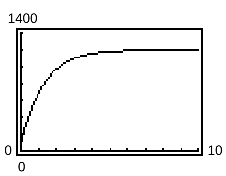
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

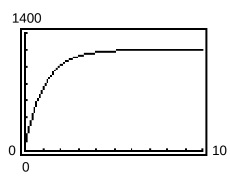
$$w = 1200 - Ce^{-kt}$$

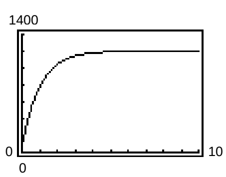
$w(0) = 60 = 1200 - C \qquad C = 1200 - 60 = 1140$

$w = 1200 - 1140e^{-kt}$

(a)







(b) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(c) Maximum weight: 1200 pounds

$\lim_{t \quad 0} w = 1200$

103. (a)
$$\frac{dv}{dt} = k(W - v)$$

$$\int \frac{dv}{W - v} = \int k\,dt$$

$$-\ln(W - v) = kt + C_1$$

$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20$, $v = 0$ when $t = 0$, and

$v = 5$ when $t = 1$.

$C = 20$, $k = -\ln(3/4)$

Particular solution:

$$v = 20(1 - e^{\ln(3/4)t}) \approx 20(1 - e^{-0.2877t})$$

(b) $s = \displaystyle\int 20(1 - e^{-0.2877t})\,dt$

$\approx 20\left[t + 3.4761e^{-0.2877t}\right] + C$

Since $S(0) = 0$, $C \approx -69.5$ and we have $s \approx 20t + 69.5(e^{-0.2877t} - 1)$.

105. Given family (circles): $x^2 + y^2 = C$

$$2x + 2yy' = 0$$

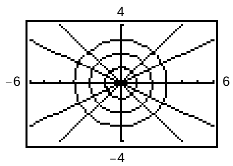
$$y' = -\frac{x}{y}$$

Orthogonal trajectory (lines): $y' = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln K$$

$$y = Kx$$



107. Given family (parabolas): $x^2 = Cy$

$$2x = Cy'$$

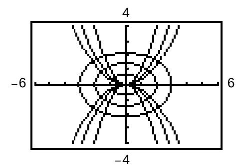
$$y' = \frac{2x}{C} = \frac{2x}{x^2/y} = \frac{2y}{x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{x}{2y}$

$$2\int y\,dy = -\int x\,dx$$

$$y^2 = -\frac{x^2}{2} + K_1$$

$$x^2 + 2y^2 = K$$



109. Given family: $y^2 = Cx^3$

$$2yy' = 3Cx^2$$

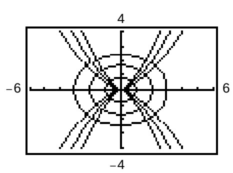
$$y' = \frac{3Cx^2}{2y} = \frac{3x^2}{2y}\left(\frac{y^2}{x^3}\right) = \frac{3y}{2x}$$

Orthogonal trajectory (ellipses): $y' = -\frac{2x}{3y}$

$$3\int y\,dy = -2\int x\,dx$$

$$\frac{3y^2}{2} = -x^2 + K_1$$

$$3y^2 + 2x^2 = K$$



111. A general solution of order n has n arbitrary constants while in a particular solution initial conditions are given in order to solve for all these constants.

113. $M(x, y)dx + N(x, y)dy = 0$, where M and N are homogeneous functions of the same degree.

115. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

117. False

$$f(tx, ty) = t^2x^2 + t^2xy + 2$$

$$\neq t^2 f(x, y)$$

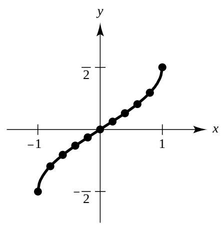
Section 5.8 Inverse Trigonometric Functions: Differentiation

1. $y = \arcsin x$

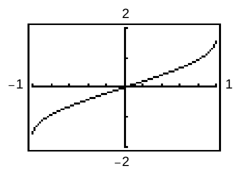
(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1.571	-0.927	-0.644	-0.412	-0.201	0	0.201	0.412	0.644	0.927	1.571

(b)



(c)



(d) Symmetric about origin:
$\arcsin(-x) = -\arcsin x$

Intercept: $(0, 0)$

3. False.

$$\arccos \frac{1}{2} = \frac{\pi}{3}$$

since the range is $[0, \pi]$.

5. $\arcsin \dfrac{1}{2} = \dfrac{\pi}{6}$

7. $\arccos \dfrac{1}{2} = \dfrac{\pi}{3}$

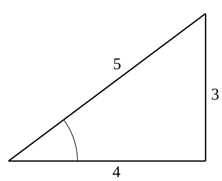
9. $\arctan \dfrac{\sqrt{3}}{3} = \dfrac{\pi}{6}$

11. $\operatorname{arccsc}(-\sqrt{2}) = -\dfrac{\pi}{4}$

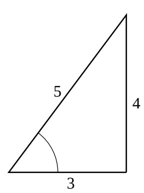
13. $\arccos(-0.8) \approx 2.50$

15. $\operatorname{arcsec}(1.269) = \arccos\left(\dfrac{1}{1.269}\right) \approx 0.66$

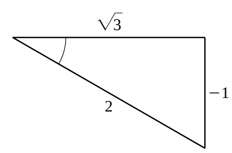
17. (a) $\sin\left(\arctan \frac{3}{4}\right) = \frac{3}{5}$



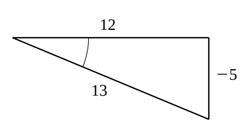
(b) $\sec\left(\arcsin \frac{4}{5}\right) = \frac{5}{3}$



19. (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right] = \cot\left(-\dfrac{\pi}{6}\right) = -\sqrt{3}$



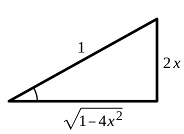
(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right] = -\dfrac{13}{5}$



21. $y = \cos(\arcsin 2x)$

$\theta = \arcsin 2x$

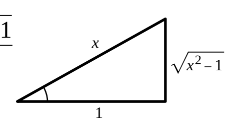
$y = \cos \theta = \sqrt{1 - 4x^2}$



23. $y = \sin(\operatorname{arcsec} x)$

$\theta = \operatorname{arcsec} x,\ 0 \quad \theta \quad \pi,\ \theta \neq \dfrac{\pi}{2}$

$y = \sin \theta = \dfrac{\sqrt{x^2 - 1}}{|x|}$

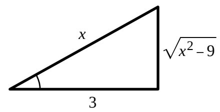


The absolute value bars on x are necessary because of the restriction $0 \quad \theta \quad \pi$, $\theta \neq \pi/2$, and $\sin \theta$ for this domain must always be nonnegative.

25. $y = \tan\left(\text{arcsec}\,\dfrac{x}{3}\right)$

$\theta = \text{arcsec}\,\dfrac{x}{3}$

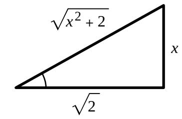
$y = \tan\theta = \dfrac{\sqrt{x^2 - 9}}{3}$



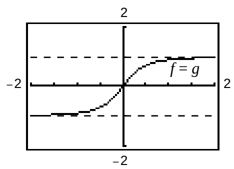
27. $y = \csc\left(\arctan\dfrac{x}{\sqrt{2}}\right)$

$\theta = \arctan\dfrac{x}{\sqrt{2}}$

$y = \csc\theta = \dfrac{\sqrt{x^2 + 2}}{x}$



29. $\sin(\arctan 2x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

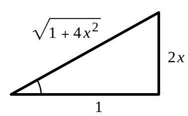


Asymptotes: $y = \pm 1$

$\arctan 2x = \theta$

$\tan\theta = 2x$

$\sin\theta = \dfrac{2x}{\sqrt{1 + 4x^2}}$



31. $\arcsin(3x - \pi) = \frac{1}{2}$

$3x - \pi = \sin\left(\frac{1}{2}\right)$

$x = \frac{1}{3}\left[\sin\left(\frac{1}{2}\right) + \pi\right] \approx 1.207$

33. $\arcsin\sqrt{2x} = \arccos\sqrt{x}$

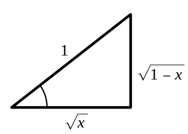
$\sqrt{2x} = \sin\left(\arccos\sqrt{x}\right)$

$\sqrt{2x} = \sqrt{1 - x}, 0 \quad x \quad 1$

$2x = 1 - x$

$3x = 1$

$x = \dfrac{1}{3}$



35. (a) $\text{arccsc}\,x = \arcsin\dfrac{1}{x}, |x| \quad 1$

Let $y = \text{arccsc}\,x$. Then for

$-\dfrac{\pi}{2} \quad y < 0 \text{ and } 0 < y \quad \dfrac{\pi}{2},$

$\csc y = x \quad \sin y = 1/x$. Thus, $y = \arcsin(1/x)$. Therefore, $\text{arccsc}\,x = \arcsin(1/x)$.

(b) $\arctan x + \arctan\dfrac{1}{x} = \dfrac{\pi}{2}, x > 0$

Let $y = \arctan x + \arctan(1/x)$. Then,

$\tan y = \dfrac{\tan(\arctan x) + \tan[\arctan(1/x)]}{1 - \tan(\arctan x)\tan[\arctan(1/x)]}$

$= \dfrac{x + (1/x)}{1 - x(1/x)}$

$= \dfrac{x + (1/x)}{0}$ (which is undefined).

Thus, $y = \pi/2$. Therefore, $\arctan x + \arctan(1/x) = \pi/2$.

37. $f(x) = \arcsin(x - 1)$

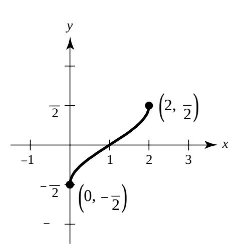
$x - 1 = \sin y$

$x = 1 + \sin y$

Domain: $[0, 2]$

Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$f(x)$ is the graph of $\arcsin x$ shifted 1 unit to the right.



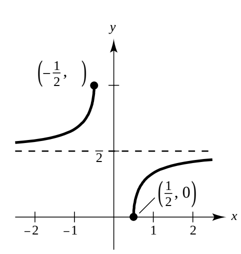
39. $f(x) = \text{arcsec}\,2x$

$2x = \sec y$

$x = \dfrac{1}{2}\sec y$

Domain: $\left(-\infty, -\dfrac{1}{2}\right], \left[\dfrac{1}{2}, \infty\right)$

Range: $\left[0, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \pi\right]$



41. $f(x) = 2\arcsin(x - 1)$

$$f'(x) = \frac{2}{\sqrt{1 - (x - 1)^2}} = \frac{2}{\sqrt{2x - x^2}}$$

43. $g(x) = 3\arccos\frac{x}{2}$

$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4 - x^2}}$$

45. $f(x) = \arctan\frac{x}{a}$

$$f'(x) = \frac{1/a}{1 + (x^2/a^2)} = \frac{a}{a^2 + x^2}$$

47. $g(x) = \frac{\arcsin 3x}{x}$

$$g'(x) = \frac{x\left(3/\sqrt{1 - 9x^2}\right) - \arcsin 3x}{x^2}$$

$$= \frac{3x - \sqrt{1 - 9x^2}\arcsin 3x}{x^2\sqrt{1 - 9x^2}}$$

49. $h(t) = \sin(\arccos t) = \sqrt{1 - t^2}$

$$h'(t) = \frac{1}{2}(1 - t^2)^{-1/2}(-2t) = \frac{-t}{\sqrt{1 - t^2}}$$

51. $y = x\arccos x - \sqrt{1 - x^2}$

$$y' = \arccos x - \frac{x}{\sqrt{1 - x^2}} - \frac{1}{2}(1 - x^2)^{-1/2}(-2x)$$

$$= \arccos x$$

53. $y = \frac{1}{2}\left(\frac{1}{2}\ln\frac{x + 1}{x - 1} + \arctan x\right)$

$$= \frac{1}{4}[\ln(x + 1) - \ln(x - 1)] + \frac{1}{2}\arctan x$$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x + 1} - \frac{1}{x - 1}\right) + \frac{1/2}{1 + x^2} = \frac{1}{1 - x^4}$$

55. $y = x\arcsin x + \sqrt{1 - x^2}$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1 - x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1 - x^2}} = \arcsin x$$

57. $y = 8\arcsin\frac{x}{4} - \frac{x\sqrt{16 - x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1 - (x/4)^2}} - \frac{\sqrt{16 - x^2}}{2} - \frac{x}{4}(16 - x^2)^{-1/2}(-2x)$$

$$= \frac{8}{\sqrt{16 - x^2}} - \frac{\sqrt{16 - x^2}}{2} + \frac{x^2}{2\sqrt{16 - x^2}}$$

$$= \frac{16 - (16 - x^2) + x^2}{2\sqrt{16 - x^2}} = \frac{x^2}{\sqrt{16 - x^2}}$$

59. $y = \arctan x + \frac{x}{1 + x^2}$

$$y' = \frac{1}{1 + x^2} + \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2}$$

$$= \frac{(1 + x^2) + (1 - x^2)}{(1 + x^2)^2}$$

$$= \frac{2}{(1 + x^2)^2}$$

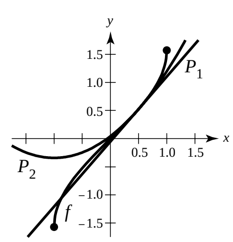
61. $f(x) = \arcsin x,\ a = \frac{1}{2}$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$



63. $f(x) = \operatorname{arcsec} x - x$

$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}} - 1$$

$$= 0 \text{ when } |x|\sqrt{x^2 - 1} = 1.$$

$$x^2(x^2 - 1) = 1$$

$$x^4 - x^2 - 1 = 0 \text{ when } x^2 = \frac{1 + \sqrt{5}}{2} \text{ or}$$

$$x = \pm\sqrt{\frac{1 + \sqrt{5}}{2}} = \pm 1.272$$

Relative maximum: $(1.272, -0.606)$

Relative minimum: $(-1.272, 3.747)$

65. $f(x) = \arctan x - \arctan(x - 4)$

$$f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + (x - 4)^2} = 0$$

$$1 + x^2 = 1 + (x - 4)^2$$

$$0 = -8x + 16$$

$$x = 2$$

By the First Derivative Test, $(2, 2.214)$ is a relative maximum.

67. The trigonometric functions are not one-to-one on $(-\infty, \infty)$, sot their domains must be restricted to intervals on which they are one-to-one.

69. $y = \operatorname{arccot} x, 0 < y < \pi$

$$x = \cot y$$

$$\tan y = \frac{1}{x}$$

So, graph the function

$$y = \arctan\left(\frac{1}{x}\right) \text{ for } x > 0 \text{ and } y = \arctan\left(\frac{1}{x}\right) + \pi \text{ for } x < 0.$$

71. (a) $\cot \theta = \dfrac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

(b) $$\frac{d\theta}{dt} = \frac{-\frac{1}{5}}{1 + \left(\frac{x}{5}\right)^2}\frac{dx}{dt} = \frac{-5}{x^2 + 25}\frac{dx}{dt}$$

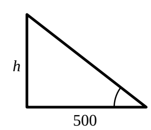
If $\dfrac{dx}{dt} = -400$ and $x = 10$, $\dfrac{d\theta}{dt} = 16$ rad/hr.

If $\dfrac{dx}{dt} = -400$ and $x = 3$, $\dfrac{d\theta}{dt} \approx 58.824$ rad/hr.

73. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec.

(b) $$\tan \theta = \frac{h}{500} = \frac{-16t^2 + 256}{500}$$



$$\theta = \arctan\left[\frac{16}{500}(-t^2 + 16)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + [(4/125)(-t^2 + 16)]^2} = \frac{-1000t}{15{,}625 + 16(16 - t^2)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

75. $\tan(\arctan x + \arctan y) = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \dfrac{x + y}{1 - xy}, xy \neq 1$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right), xy \neq 1.$$

Let $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\frac{(1/2) + (1/3)}{1 - [(1/2)\cdot(1/3)]} = \arctan\frac{5/6}{1 - (1/6)} = \arctan\frac{5/6}{5/6} = \arctan 1 = \frac{\pi}{4}$$

77. $f(x) = kx + \sin x$

$f'(x) = k + \cos x \quad 0 \text{ for } k \quad 1$

$f'(x) = k + \cos x \quad 0 \text{ for } k \quad -1$

Therefore, $f(x) = kx + \sin x$ is strictly monotonic and has an inverse for $k \quad -1$ or $k \quad 1$.

79. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

81. True

$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

Section 5.9 Inverse Trigonometric Functions: Integration

1. $\displaystyle\int \frac{5}{\sqrt{9 - x^2}}\,dx = 5\arcsin\left(\frac{x}{3}\right) + C$

3. Let $u = 3x$, $du = 3\,dx$.

$$\int_0^{1/6} \frac{1}{\sqrt{1 - 9x^2}}\,dx = \frac{1}{3}\int_0^{1/6} \frac{1}{\sqrt{1 - (3x)^2}}(3)\,dx = \left[\frac{1}{3}\arcsin(3x)\right]_0^{1/6} = \frac{\pi}{18}$$

5. $\displaystyle\int \frac{7}{16 + x^2}\,dx = \frac{7}{4}\arctan\left(\frac{x}{4}\right) + C$

7. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{3}/2} \frac{1}{1 + 4x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{2}{1 + (2x)^2}\,dx = \left[\frac{1}{2}\arctan(2x)\right]_0^{\sqrt{3}/2} = \frac{\pi}{6}$$

9. $\displaystyle\int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx = \operatorname{arcsec}|2x| + C$

11. $\displaystyle\int \frac{x^3}{x^2 + 1}\,dx = \int\left[x - \frac{x}{x^2 + 1}\right]dx = \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + C$ (Use long division.)

13. $\displaystyle\int \frac{1}{\sqrt{1 - (x + 1)^2}}\,dx = \arcsin(x + 1) + C$

15. Let $u = t^2$, $du = 2t\,dt$.

$$\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{1}{\sqrt{1 - (t^2)^2}}(2t)\,dt = \frac{1}{2}\arcsin(t^2) + C$$

17. Let $u = \arcsin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$.

$$\int_0^{1/\sqrt{2}} \frac{\arcsin}{\sqrt{1-x^2}}\,dx = \left[\frac{1}{2}\arcsin^2 x\right]_0^{1/\sqrt{2}} = \frac{\pi^2}{32} \approx 0.308$$

19. Let $u = 1 - x^2$, $du = -2x\,dx$.

$$\int_{-1/2}^{0} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_{-1/2}^{0}(1-x^2)^{-1/2}(-2x)\,dx$$

$$= \left[-\sqrt{1-x^2}\right]_{-1/2}^{0} = \frac{\sqrt{3}-2}{2}$$

$$\approx -0.134$$

21. Let $u = e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{4+(e^{2x})^2}\,dx = \frac{1}{4}\arctan\frac{e^{2x}}{2} + C$$

23. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_{\pi/2}^{\pi}\frac{-\sin x}{1+\cos^2 x}\,dx$$

$$= \left[-\arctan(\cos x)\right]_{\pi/2}^{\pi} = \frac{\pi}{4}$$

25. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx.$ $u = \sqrt{x}$, $x - u^2$, $dx = 2u\,du$

$$\int \frac{1}{u\sqrt{1-u^2}}(2u\,du) = 2\int\frac{du}{\sqrt{1-u^2}} = 2\arcsin u + C$$

$$= 2\arcsin\sqrt{x} + C$$

27. $$\int \frac{x-3}{x^2+1}\,dx = \frac{1}{2}\int\frac{2x}{x^2+1}\,dx - 3\int\frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) - 3\arctan x + C$$

29. $$\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx = \int\frac{(x-3)}{\sqrt{9-(x-3)^2}}\,dx + \int\frac{8}{\sqrt{9-(x-3)^2}}\,dx$$

$$= -\sqrt{9-(x-3)^2} - 8\arcsin\left(\frac{x-3}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 8\arcsin\left(\frac{x}{3}-1\right) + C$$

31. $$\int_0^2 \frac{1}{x^2-2x+2}\,dx = \int_0^2\frac{1}{1+(x-1)^2}\,dx = \left[\arctan(x-1)\right]_0^2 = \frac{\pi}{2}$$

33. $$\int \frac{2x}{x^2+6x+13}\,dx = \int\frac{2x+6}{x^2+6x+13}dx - 6\int\frac{1}{x^2+6x+13}\,dx = \int\frac{2x+6}{x^2+6x+13}\,dx - 6\int\frac{1}{4+(x+3)^2}\,dx$$

$$= \ln|x^2+6x+13| - 3\arctan\left(\frac{x+3}{2}\right) + C$$

35. $$\int \frac{1}{\sqrt{-x^2-4x}}\,dx = \int\frac{1}{\sqrt{4-(x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{2}\right) + C$$

37. Let $u = -x^2 - 4x$, $du = (-2x-4)\,dx$.

$$\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx = -\frac{1}{2}\int(-x^2-4x)^{-1/2}(-2x-4)\,dx = -\sqrt{-x^2-4x} + C$$

39. $$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int_2^3\frac{2x-4}{\sqrt{4x-x^2}}\,dx + \int_2^3\frac{1}{\sqrt{4x-x^2}}\,dx = -\int_2^3(4x-x^2)^{-1/2}(4-2x)\,dx + \int_2^3\frac{1}{\sqrt{4-(x-2)^2}}\,dx$$

$$= \left[-2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\right]_2^3 = 4 - 2\sqrt{3} + \frac{\pi}{6} \approx 1.059$$

41. Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 2x^2 + 2}\,dx = \frac{1}{2}\int \frac{2x}{(x^2 + 1)^2 + 1}\,dx = \frac{1}{2}\arctan(x^2 + 1) + C$$

43. Let $u = \sqrt{e^t - 3}$. Then $u^2 + 3 = e^t$, $2u\,du = e^t\,dt$, and $\dfrac{2u\,du}{u^2 + 3} = dt$.

$$\int \sqrt{e^t - 3}\,dt = \int \frac{2u^2}{u^2 + 3}\,du = \int 2\,du - \int 6\,\frac{1}{u^2 + 3}\,du$$

$$= 2u - 2\sqrt{3}\arctan\frac{u}{\sqrt{3}} + C = 2\sqrt{e^t - 3} - 2\sqrt{3}\arctan\sqrt{\frac{e^t - 3}{3}} + C$$

45. A perfect square trinomial is an expression in x with three terms that factor as a perfect square.

Example: $x^2 + 6x + 9 = (x + 3)^2$

47. (a) $\displaystyle\int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin x + C,\ u = x$

(b) $\displaystyle\int \frac{x}{\sqrt{1 - x^2}}\,dx = -\sqrt{1 - x^2} + C,\ u = 1 - x^2$

(c) $\displaystyle\int \frac{1}{x\sqrt{1 - x^2}}\,dx$ cannot be evaluated using the basic integration rules.

49. (a) $\displaystyle\int \sqrt{x - 1}\,dx = \frac{2}{3}(x - 1)^{3/2} + C,\ u = x - 1$

(b) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int x\sqrt{x - 1}\,dx = \int (u^2 + 1)(u)(2u)\,du = 2\int (u^4 + u^2)\,du = 2\left(\frac{u^5}{5} + \frac{u^3}{3}\right) + C$$

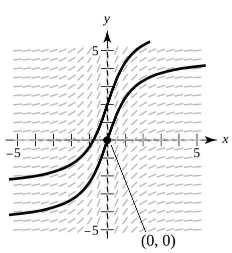
$$= \frac{2}{15}u^3(3u^2 + 5) + C = \frac{2}{15}(x - 1)^{3/2}[3(x - 1) + 5] + C = \frac{2}{15}(x - 1)^{3/2}(3x + 2) + C$$

(c) Let $u = \sqrt{x - 1}$. Then $x = u^2 + 1$ and $dx = 2u\,du$.

$$\int \frac{x}{\sqrt{x - 1}}\,dx = \int \frac{u^2 + 1}{u}(2u)\,du = 2\int (u^2 + 1)\,du = 2\left(\frac{u^3}{3} + u\right) + C = \frac{2}{3}u(u^2 + 3) + C = \frac{2}{3}\sqrt{x - 1}(x + 2) + C$$

Note: In (b) and (c), substitution was necessary *before* the basic integration rules could be used.

51. (a)

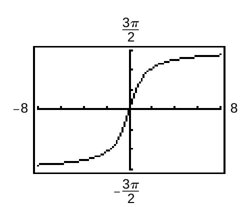


(b) $\dfrac{dy}{dx} = \dfrac{3}{1 + x^2}$, $(0, 0)$

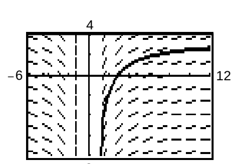
$$y = 3\int \frac{dx}{1 + x^2} = 3\arctan x + C$$

$(0, 0)$: $0 = 3\arctan(0) + C \qquad C = 0$

$$y = 3\arctan x$$



53. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2 - 1}}$, $y(3) = 0$

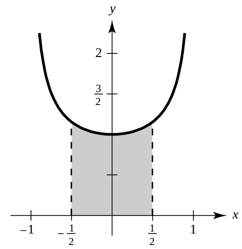


55. $\displaystyle A = \int_1^3 \frac{1}{x^2 - 2x + 1 + 4}\,dx = \int_1^3 \frac{1}{(x - 1)^2 + 2^2}\,dx$

$$= \left[\frac{1}{2}\arctan\left(\frac{x - 1}{2}\right)\right]_1^3 = \frac{1}{2}\arctan(1) = \frac{\pi}{8} \approx 0.3927$$

57. Area $\approx (1)(1) = 1$

Matches (c)



59. (a) $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx = \Big[4\arctan x\Big]_0^1 = 4\arctan 1 - 4\arctan 0 = 4\left(\frac{\pi}{4}\right) - 4(0) = \pi$

(b) Let $n = 6$.

$$4\int_0^1 \frac{4}{1+x^2}\,dx \approx 4\left(\frac{1}{36}\right)\left[1 + \frac{4}{1+(1/36)} + \frac{2}{1+(1/9)} + \frac{4}{1+(1/4)} + \frac{2}{1+(4/9)} + \frac{4}{1+(25/36)} + \frac{1}{2}\right] \approx 3.1415918$$

(c) 3.1415927

61. (a) $\displaystyle\frac{d}{dx}\left[\arcsin\left(\frac{u}{a}\right) + C\right] = \frac{1}{\sqrt{1-(u^2/a^2)}}\left(\frac{u'}{a}\right) = \frac{u'}{\sqrt{a^2-u^2}}$

Thus, $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C.$

(b) $\displaystyle\frac{d}{dx}\left[\frac{1}{a}\arctan\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{1+(u/a)^2}\right] = \frac{1}{a^2}\left[\frac{u'}{(a^2+u^2)/a^2}\right] = \frac{u'}{a^2+u^2}$

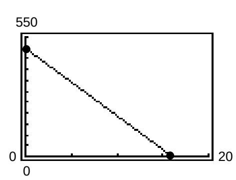
Thus, $\displaystyle\int \frac{du}{a^2+u^2} = \int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C.$

(c) Assume $u > 0$.

$\displaystyle\frac{d}{dx}\left[\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C\right] = \frac{1}{a}\left[\frac{u'/a}{(u/a)\sqrt{(u/a)^2-1}}\right] = \frac{1}{a}\left[\frac{u'}{u\sqrt{u^2-a^2)/a^2}}\right] = \frac{u'}{u\sqrt{u^2-a^2}}$. The case $u < 0$ is handled in a similar manner.

Thus, $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \int \frac{u'}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\,\text{arsec}\,\frac{|u|}{a} + C.$

63. (a) $v(t) = -32t + 500$



(b) $\displaystyle s(t) = \int v(t)\,dt = \int(-32t + 500)\,dt$

$= -16t^2 + 500t + C$

$s(0) = -16(0) + 500(0) + C = 0 \qquad C = 0$

$s(t) = -16t^2 + 500t$

When the object reaches its maximum height, $v(t) = 0$.

$v(t) = -32t + 500 = 0$

$-32t = -500$

$t = 15.625$

$s(15.625) = -16(15.625)^2 + 500(15.625)$

$= 3906.25$ ft (Maximum height)

—CONTINUED—

63. —CONTINUED—

(c) $$\int \frac{1}{32 + kv^2}\,dv = -\int dt$$

$$\frac{1}{\sqrt{32k}} \arctan\left(\sqrt{\frac{k}{32}}v\right) = -t + C_1$$

$$\arctan\left(\sqrt{\frac{k}{32}}v\right) = -\sqrt{32k}\,t + C$$

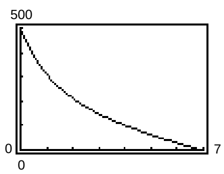
$$\sqrt{\frac{k}{32}}v = \tan\left(C - \sqrt{32k}\,t\right)$$

$$v = \sqrt{\frac{32}{k}}\tan\left(C - \sqrt{32k}\,t\right)$$

When $t = 0$, $v = 500$, $C = \arctan\left(500\sqrt{k/32}\right)$, and we have

$$v(t) = \sqrt{\frac{32}{k}}\tan\left[\arctan\left(500\sqrt{\frac{k}{32}}\right) - \sqrt{32k}\,t\right].$$

(d) When $k = 0.001$, $v(t) = \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right]$.



$v(t) = 0$ when $t_0 \approx 6.86$ sec.

(e) $$h = \int_0^{6.86} \sqrt{32{,}000}\tan\left[\arctan\left(500\sqrt{0.00003125}\right) - \sqrt{0.032}\,t\right] dt$$

Simpson's Rule: $n = 10$; $h \approx 1088$ feet

(f) Air resistance lowers the maximum height.

Section 5.10 Hyperbolic Functions

1. (a) $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.018$

(b) $\tanh(-2) = \dfrac{\sinh(-2)}{\cosh(-2)} = \dfrac{e^{-2} - e^2}{e^{-2} + e^2} \approx -0.964$

3. (a) $\operatorname{csch}(\ln 2) = \dfrac{2}{e^{\ln 2} - e^{-\ln 2}} = \dfrac{2}{2 - (1/2)} = \dfrac{4}{3}$

(b) $\coth(\ln 5) = \dfrac{\cosh(\ln 5)}{\sinh(\ln 5)} = \dfrac{e^{\ln 5} + e^{-\ln 5}}{e^{\ln 5} - e^{-\ln 5}}$

$$= \frac{5 + (1/5)}{5 - (1/5)} = \frac{13}{12}$$

5. (a) $\cosh^{-1}(2) = \ln\left(2 + \sqrt{3}\right) \approx 1.317$

(b) $\operatorname{sech}^{-1}\left(\dfrac{2}{3}\right) = \ln\left(\dfrac{1 + \sqrt{1 - (4/9)}}{2/3}\right) \approx 0.962$

7. $$\tanh^2 x + \operatorname{sech}^2 x = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x} + 4}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1$$

9. $\sinh x \cosh y + \cosh x \sinh y = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$

$$= \frac{1}{4}[e^{x+y} - e^{-x+y} + e^{x-y} - e^{-(x+y)} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-(x+y)}]$$

$$= \frac{1}{4}[2(e^{x+y} - e^{-(x+y)})] = \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x + y)$$

11. $3 \sinh x + 4 \sinh^3 x = \sinh x(3 + 4 \sinh^2 x) = \left(\frac{e^x - e^{-x}}{2}\right)\left[3 + 4\left(\frac{e^x - e^{-x}}{2}\right)^2\right]$

$$= \left(\frac{e^x - e^{-x}}{2}\right)[3 + e^{2x} - 2 + e^{-2x}] = \frac{1}{2}(e^x - e^{-x})(e^{2x} + e^{-2x} + 1)$$

$$= \frac{1}{2}[e^{3x} + e^{-x} + e^x - e^x - e^{-3x} - e^{-x}] = \frac{e^{3x} - e^{-3x}}{2} = \sinh(3x)$$

13. $\sinh x = \frac{3}{2}$

$\cosh^2 x - \left(\frac{3}{2}\right)^2 = 1 \qquad \cosh^2 x = \frac{13}{4} \qquad \cosh x = \frac{\sqrt{13}}{2}$

$$\tanh x = \frac{3/2}{\sqrt{13}/2} = \frac{3\sqrt{13}}{13}$$

$$\operatorname{csch} x = \frac{1}{3/2} = \frac{2}{3}$$

$$\operatorname{sech} x = \frac{1}{\sqrt{13}/2} = \frac{2\sqrt{13}}{13}$$

$$\coth x = \frac{1}{3/\sqrt{13}} = \frac{\sqrt{13}}{3}$$

15. $y = \sinh(1 - x^2)$

$y' = -2x \cosh(1 - x^2)$

17. $f(x) = \ln(\sinh x)$

$f'(x) = \frac{1}{\sinh}(\cosh x) = \coth x$

19. $y = \ln\left(\tanh \frac{x}{2}\right)$

$$y' = \frac{1/2}{\tanh(x/2)} \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{1}{2 \sinh(x/2) \cosh(x/2)}$$

$$= \frac{1}{\sinh x} = \operatorname{csch} x$$

21. $h(x) = \frac{1}{4} \sinh(2x) - \frac{x}{2}$

$h'(x) = \frac{1}{2} \cosh(2x) - \frac{1}{2} = \frac{\cosh(2x) - 1}{2} = \sinh^2 x$

23. $f(t) = \arctan(\sinh t)$

$$f'(t) = \frac{1}{1 + \sinh^2 t}(\cosh t)$$

$$= \frac{\cosh t}{\cosh^2 t} = \operatorname{sech} t$$

25. Let $y = g(x)$.

$$y = x^{\cosh x}$$

$$\ln y = \cosh x \ln x$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{\cosh x}{x} + \sinh x \ln x$$

$$\frac{dy}{dx} = \frac{y}{x}[\cosh x + x(\sinh x) \ln x]$$

$$= \frac{x^{\cosh x}}{x}[\cosh x + x(\sinh x) \ln x]$$

27. $y = (\cosh x - \sinh x)^2$

$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$

$= -2(\cosh x - \sinh x)^2 = -2e^{-2x}$

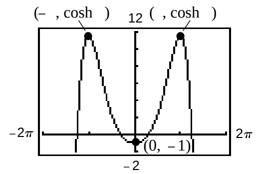
29. $f(x) = \sin x \sinh x - \cos x \cosh x, -4 \quad x \quad 4$

$f'(x) = \sin x \cosh x + \cos x \sinh x - \cos x \sinh x + \sin x \cosh x$

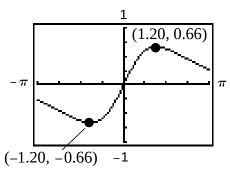
$= 2 \sin x \cosh x = 0$ when $x = 0, \pm\pi$.

Relative maxima: $(\pm\pi, \cosh \pi)$

Relative minimum: $(0, -1)$



31. $g(x) = x \operatorname{sech} x = \dfrac{x}{\cosh x}$



Relative maximum: $(1.20, 0.66)$

Relative minimum: $(-1.20, -0.66)$

33. $y = a \sinh x$

$y' = a \cosh x$

$y'' = a \sinh x$

$y''' = a \cosh x$

Therefore, $y''' - y' = 0$.

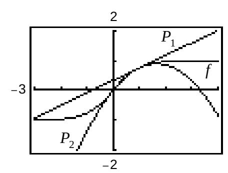
35. $f(x) = \tanh x$ $\qquad f(1) = \tanh(1) \approx 0.7616$

$f'(x) = \operatorname{sech}^2 x$ $\qquad f'(1) = \dfrac{1}{\cosh^2(1)} \approx 0.4200$

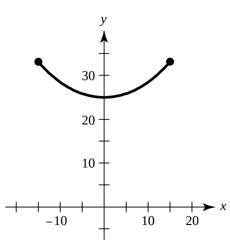
$f''(x) = -2 \operatorname{sech}^2 x \cdot \tanh x$ $\qquad f''(1) \approx -0.6397$

$P_1(x) = f(1) + f'(1)(x - 1) = 0.7616 + 0.42(x - 1)$

$P_2(x) = 0.7616 + 0.42(x - 1) - \dfrac{0.6397}{2}(x - 1)^2$



37. (a) $y = 10 + 15 \cosh \dfrac{x}{15}, -15 \quad x \quad 15$



(b) At $x = \pm 15$, $y = 10 + 15 \cosh(1) \approx 33.146$.

At $x = 0$, $y = 10 + 15 \cosh(1) = 25$.

(c) $y' = \sinh \dfrac{x}{15}$. At $x = 15$, $y' = \sinh(1) \approx 1.175$

39. Let $u = 1 - 2x$, $du = -2\, dx$.

$$\int \sinh(1 - 2x)\, dx = -\frac{1}{2}\int \sinh(1 - 2x)(-2)\, dx$$

$$= -\frac{1}{2} \cosh(1 - 2x) + C$$

41. Let $u = \cosh(x - 1)$, $du = \sinh(x - 1)\, dx$.

$$\int \cosh^2(x - 1) \sinh(x - 1)\, dx = \frac{1}{3} \cosh^3(x - 1) + C$$

43. Let $u = \sinh x$, $du = \cosh x\,dx$.

$$\int \frac{\cosh x}{\sinh x}\,dx = \ln|\sinh x| + C$$

45. Let $u = \frac{x^2}{2}$, $du = x\,dx$.

$$\int x \operatorname{csch}^2 \frac{x^2}{2}\,dx = \int \left(\operatorname{csch}^2 \frac{x^2}{2}\right) x\,dx = -\coth \frac{x^2}{2} + C$$

47. Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$.

$$\int \frac{\operatorname{csch}(1/x)\coth(1/x)}{x^2}\,dx = -\int \operatorname{csch}\frac{1}{x}\coth\frac{1}{x}\left(-\frac{1}{x^2}\right)dx = \operatorname{csch}\frac{1}{x} + C$$

49. $$\int_0^4 \frac{1}{25 - x^2}\,dx = \left[\frac{1}{10}\ln\left|\frac{5 + x}{5 - x}\right|\right]_0^4 = \frac{1}{10}\ln 9 = \frac{1}{5}\ln 3$$

51. Let $u = 2x$, $du = 2\,dx$.

$$\int_0^{\sqrt{2}/4} \frac{1}{\sqrt{1 - (2x)^2}}(2)\,dx = \Big[\arcsin(2x)\Big]_0^{\sqrt{2}/4} = \frac{\pi}{4}$$

53. Let $u = x^2$, $du = 2x\,dx$.

$$\int \frac{x}{x^4 + 1}\,dx = \frac{1}{2}\int \frac{2x}{(x^2)^2 + 1}\,dx = \frac{1}{2}\arctan(x^2) + C$$

55. $y = \cosh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{9x^2 - 1}}$$

57. $y = \sinh^{-1}(\tan x)$

$$y' = \frac{1}{\sqrt{\tan^2 x + 1}}(\sec^2 x) = |\sec x|$$

59. $y = \coth^{-1}(\sin 2x)$

$$y' = \frac{1}{1 - \sin^2 2x}(2\cos 2x) = 2\sec 2x$$

61. $y = 2x\sinh^{-1}(2x) - \sqrt{1 + 4x^2}$

$$y' = 2x\left(\frac{2}{\sqrt{1 + 4x^2}}\right) + 2\sinh^{-1}(2x) - \frac{4x}{\sqrt{1 + 4x^2}} = 2\sinh^{-1}(2x)$$

63. See page 395.

65. $y = a\operatorname{sech}^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$

$$\frac{dy}{dx} = \frac{-1}{(x/a)\sqrt{1 - (x^2/a^2)}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{-a^2}{x\sqrt{a^2 - x^2}} + \frac{x}{\sqrt{a^2 - x^2}} = \frac{x^2 - a^2}{x\sqrt{a^2 - x^2}} = \frac{-\sqrt{a^2 - x^2}}{x}$$

67. $$\int \frac{1}{\sqrt{1 + e^{2x}}}\,dx = \int \frac{e^x}{e^x\sqrt{1 + (e^x)^2}}\,dx = -\operatorname{csch}^{-1}(e^x) + C = -\ln\left(\frac{1 + \sqrt{1 + e^{2x}}}{e^x}\right) + C$$

69. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{1}{\sqrt{x}\sqrt{1 + x}}\,dx = 2\int \frac{1}{\sqrt{1 + \left(\sqrt{x}\right)^2}}\left(\frac{1}{2\sqrt{x}}\right)dx = 2\sinh^{-1}\sqrt{x} + C = 2\ln\left(\sqrt{x} + \sqrt{1 + x}\right) + C$$

71. $$\int \frac{-1}{4x - x^2}\,dx = \int \frac{1}{(x - 2)^2 - 4}\,dx = \frac{1}{4}\ln\left|\frac{(x - 2) - 2}{(x - 2) + 2}\right| = \frac{1}{4}\ln\left|\frac{x - 4}{x}\right| + C$$

73. $\displaystyle\int \frac{1}{1-4x-2x^2}\,dx = \int \frac{1}{3-2(x+1)^2}\,dx = \frac{-1}{\sqrt{2}}\int \frac{\sqrt{2}}{\left[\sqrt{2}(x+1)\right]^2-\left(\sqrt{3}\right)^2}\,dx$

$$= \frac{-1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}(x+1)-\sqrt{3}}{\sqrt{2}(x+1)+\sqrt{3}}\right| + C = \frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}(x+1)+\sqrt{3}}{\sqrt{2}(x+1)-\sqrt{3}}\right| + C$$

75. Let $u = 4x - 1$, $du = 4\,dx$.

$$y = \int \frac{1}{\sqrt{80+8x-16x^2}}\,dx = \frac{1}{4}\int \frac{4}{\sqrt{81-(4x-1)^2}}\,dx = \frac{1}{4}\arcsin\left(\frac{4x-1}{9}\right) + C$$

77. $\displaystyle y = \int \frac{x^3-21x}{5+4x-x^2}\,dx = \int\left(-x-4+\frac{20}{5+4x-x^2}\right)dx = \int(-x-4)\,dx + 20\int\frac{1}{3^2-(x-2)^2}\,dx$

$$= -\frac{x^2}{2} - 4x + \frac{20}{6}\ln\left|\frac{(x-2)+3}{(x-2)-3}\right| + C = -\frac{x^2}{2} - 4x + \frac{10}{3}\ln\left|\frac{x+1}{x-5}\right| + C = \frac{-x^2}{2} - 4x - \frac{10}{3}\ln\left|\frac{x-5}{x+1}\right| + C$$

79. $\displaystyle A = 2\int_0^4 \operatorname{sech}\frac{x}{2}\,dx$

$$= 2\int_0^4 \frac{2}{e^{x/2}+e^{-x/2}}\,dx$$

$$= 4\int_0^4 \frac{e^{x/2}}{(e^{x/2})^2+1}\,dx$$

$$= \left[8\arctan(e^{x/2})\right]_0^4$$

$$= 8\arctan(e^2) - 2\pi \approx 5.207$$

81. $\displaystyle A = \int_0^2 \frac{5x}{\sqrt{x^4+1}}\,dx$

$$= \frac{5}{2}\int_0^2 \frac{2x}{\sqrt{(x^2)^2+1}}\,dx$$

$$= \left[\frac{5}{2}\ln\left(x^2+\sqrt{x^4+1}\right)\right]_0^2$$

$$= \frac{5}{2}\ln\left(4+\sqrt{17}\right) \approx 5.237$$

83. $\displaystyle\int \frac{3k}{16}\,dt = \int \frac{1}{x^2-12x+32}\,dx$

$$\frac{3kt}{16} = \int \frac{1}{(x-6)^2-4}\,dx = \frac{1}{2(2)}\ln\left|\frac{(x-6)-2}{(x-6)+2}\right| + C = \frac{1}{4}\ln\left|\frac{x-8}{x-4}\right| + C$$

When $x = 0$: $t = 0$

$$C = -\frac{1}{4}\ln(2)$$

When $x = 1$: $t = 10$

$$\frac{30k}{16} = \frac{1}{4}\ln\left|\frac{-7}{-3}\right| - \frac{1}{4}\ln(2) = \frac{1}{4}\ln\left(\frac{7}{6}\right)$$

$$k = \frac{2}{15}\ln\left(\frac{7}{6}\right)$$

When $t = 20$: $\displaystyle\left(\frac{3}{16}\right)\left(\frac{2}{15}\right)\ln\left(\frac{7}{6}\right)(20) = \frac{1}{4}\ln\frac{x-8}{2x-8}$

$$\ln\left(\frac{7}{6}\right)^2 = \ln\frac{x-8}{2x-8}$$

$$\frac{49}{36} = \frac{x-8}{2x-8}$$

$$62x = 104$$

$$x = \frac{104}{62} = \frac{52}{31} \approx 1.677 \text{ kg}$$

85. As k increases, the time required for the object to reach the ground increases.

87. $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

89. $y = \cosh^{-1} x$

$$\cosh y = x$$

$$(\sinh y)(y') = 1$$

$$y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

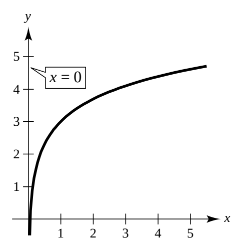
91. $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$$y' = -2(e^x + e^{-x})^{-2}(e^x - e^{-x}) = \left(\frac{-2}{e^x + e^{-x}}\right)\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -\operatorname{sech} x \tanh x$$

Review Exercises for Chapter 5

1. $f(x) = \ln x + 3$

Vertical shift 3 units upward
Vertical asymptote: $x = 0$



3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}} = \dfrac{1}{5} \ln \dfrac{(2x - 1)(2x + 1)}{4x^2 + 1} = \dfrac{1}{5}[\ln(2x - 1) + \ln(2x + 1) - \ln(4x^2 + 1)]$

5. $\ln 3 + \dfrac{1}{3}\ln(4 - x^2) - \ln x = \ln 3 + \ln \sqrt[3]{4 - x^2} - \ln x = \ln\left(\dfrac{3\sqrt[3]{4 - x^2}}{x}\right)$

7. $\ln \sqrt{x + 1} = 2$

$$\sqrt{x + 1} = e^2$$

$$x + 1 = e^4$$

$$x = e^4 - 1 \approx 53.598$$

9. $g(x) = \ln \sqrt{x} = \dfrac{1}{2} \ln x$

$$g'(x) = \frac{1}{2x}$$

11. $f(x) = x\sqrt{\ln x}$

$$f'(x) = \left(\frac{x}{2}\right)(\ln x)^{-1/2}\left(\frac{1}{x}\right) + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

13. $y = \dfrac{1}{b^2}\left[\ln(a + bx) + \dfrac{a}{a + bx}\right]$

$$\frac{dy}{dx} = \frac{1}{b^2}\left[\frac{b}{a + bx} - \frac{ab}{(a + bx)^2}\right] = \frac{x}{(a + bx)^2}$$

15. $y = -\dfrac{1}{a} \ln\left(\dfrac{a + bx}{x}\right) = -\dfrac{1}{a}[\ln(a + bx) - \ln x]$

$$\frac{dy}{dx} = -\frac{1}{a}\left(\frac{b}{a + bx} - \frac{1}{x}\right) = \frac{1}{x(a + bx)}$$

17. $u = 7x - 2,\ du = 7\,dx$

$$\int \frac{1}{7x - 2}\,dx = \frac{1}{7}\int \frac{1}{7x - 2}(7)\,dx = \frac{1}{7}\ln|7x - 2| + C$$

19. $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx = -\int \frac{-\sin x}{1 + \cos x}\,dx$

$\displaystyle = -\ln|1 + \cos x| + C$

21. $\displaystyle\int_1^4 \frac{x+1}{x}\,dx = \int_1^4 \left(1 + \frac{1}{x}\right) dx = \Big[x + \ln|x|\Big]_1^4 = 3 + \ln 4$

23. $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = \ln\left(2 + \sqrt{3}\right)$

25. (a) $f(x) = \frac{1}{2}x - 3$

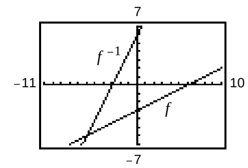
$y = \frac{1}{2}x - 3$

$2(y + 3) = x$

$2(x + 3) = y$

$f^{-1}(x) = 2x + 6$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\frac{1}{2}x - 3\right) = 2\left(\frac{1}{2}x - 3\right) + 6 = x$

$f(f^{-1}(x)) = f(2x + 6) = \frac{1}{2}(2x + 6) - 3 = x$

27. (a) $f(x) = \sqrt{x + 1}$

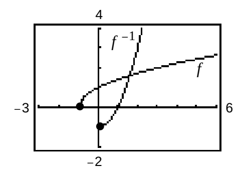
$y = \sqrt{x + 1}$

$y^2 - 1 = x$

$x^2 - 1 = y$

$f^{-1}(x) = x^2 - 1, x \quad 0$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt{x + 1}\right) = \sqrt{(x^2 - 1)^2} - 1 = x$

$f(f^{-1}(x)) = f(x^2 - 1) = \sqrt{(x^2 - 1) + 1}$

$= \sqrt{x^2} = x$ for $x \quad 0.$

29. (a) $f(x) = \sqrt[3]{x + 1}$

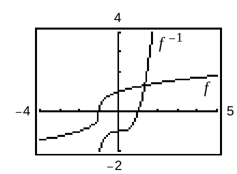
$y = \sqrt[3]{x + 1}$

$y^3 - 1 = x$

$x^3 - 1 = y$

$f^{-1}(x) = x^3 - 1$

(b)



(c) $f^{-1}(f(x)) = f^{-1}\left(\sqrt[3]{x + 1}\right) = \left(\sqrt[3]{x + 1}\right)^3 - 1 = x$

$f(f^{-1}(x)) = f(x^3 - 1) = \sqrt[3]{(x^3 - 1) + 1} = x$

31. $f(x) = x^3 + 2$

$f^{-1}(x) = (x - 2)^{1/3}$

$\displaystyle (f^{-1})'(x) = \frac{1}{3}(x - 2)^{-2/3}$

$\displaystyle (f^{-1})'(-1) = \frac{1}{3}(-1 - 2)^{-2/3} = \frac{1}{3(-3)^{2/3}}$

$\displaystyle = \frac{1}{3^{5/3}} \approx 0.160$

33. $f(x) = \tan x$

$\displaystyle f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

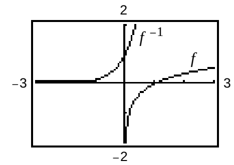
$f'(x) = \sec^2 x$

$\displaystyle f'\left(\frac{\pi}{6}\right) = \frac{4}{3}$

$\displaystyle (f^{-1})'\left(\frac{\sqrt{3}}{3}\right) = \frac{1}{f'(\pi/6)} = \frac{3}{4}$

35. (a)
$$\begin{aligned} f(x) &= \ln\sqrt{x} \\ y &= \ln\sqrt{x} \\ e^y &= \sqrt{x} \\ e^{2y} &= x \\ e^{2x} &= y \\ f^{-1}(x) &= e^{2x} \end{aligned}$$

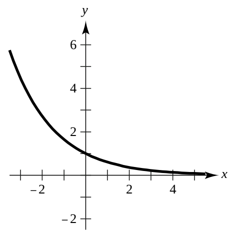
(b)



(c) $f^{-1}(f(x)) = f^{-1}(\ln\sqrt{x}) = e^{2\ln\sqrt{x}} = e^{\ln x} = x$

$f(f^{-1}(x)) = f(e^{2x}) = \ln\sqrt{e^{2x}} = \ln e^x = x$

37. $y = e^{-x/2}$



39. $f(x) = \ln(e^{-x^2}) = -x^2$

$f'(x) = -2x$

41. $g(t) = t^2e^t$

$g'(x) = t^2e^t + 2te^t = te^t(t + 2)$

43. $y = \sqrt{e^{2x} + e^{-2x}}$

$$y' = \frac{1}{2}(e^{2x} + e^{-2x})^{-1/2}(2e^{2x} - 2e^{-2x}) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}$$

45. $g(x) = \dfrac{x^2}{e^x}$

$$g'(x) = \frac{e^x(2x) - x^2e^x}{e^{2x}} = \frac{x(2-x)}{e^x}$$

47.
$$\begin{aligned} y(\ln x) + y^2 &= 0 \\ y\left(\frac{1}{x}\right) + (\ln x)\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) &= 0 \\ (2y + \ln x)\frac{dy}{dx} &= \frac{-y}{x} \\ \frac{dy}{dx} &= \frac{-y}{x(2y + \ln x)} \end{aligned}$$

49. Let $u = -3x^2$, $du = -6x\,dx$.

$$\int xe^{-3x^2}\,dx = -\frac{1}{6}\int e^{-3x^2}(-6x)\,dx = -\frac{1}{6}e^{-3x^2} + C$$

51.
$$\begin{aligned} \int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx &= \int (e^{3x} - e^x + e^{-x})\,dx \\ &= \frac{1}{3}e^{3x} - e^x - e^{-x} + C \\ &= \frac{e^{4x} - 3e^{2x} - 3}{3e^x} + C \end{aligned}$$

53.
$$\begin{aligned} \int xe^{1-x^2}\,dx &= -\frac{1}{2}\int e^{1-x^2}(-2x)\,dx \\ &= -\frac{1}{2}e^{1-x^2} + C \end{aligned}$$

55. Let $u = e^x - 1$, $du = e^x\,dx$.

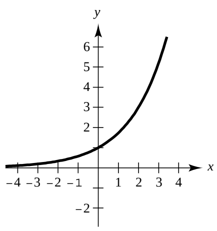
$$\int \frac{e^x}{e^x - 1}\,dx = \ln|e^x - 1| + C$$

57.
$$\begin{aligned} y &= e^x(a\cos 3x + b\sin 3x) \\ y' &= e^x(-3a\sin 3x + 3b\cos 3x) + e^x(a\cos 3x + b\sin 3x) \\ &= e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x] \\ y'' &= e^x[3(-3a + b)\cos 3x - 3(a + 3b)\sin 3x] + e^x[(-3a + b)\sin 3x + (a + 3b)\cos 3x] \\ &= e^x[(-6a - 8b)\sin 3x + (-8a + 6b)\cos 3x] \end{aligned}$$

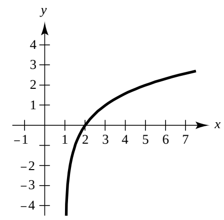
$$y'' - 2y' + 10y = e^x\{[(-6a - 8b) - 2(-3a + b) + 10b]\sin 3x + [(-8a + 6b) - 2(a + 3b) + 10a]\cos 3x\} = 0$$

59. $\text{Area} = \int_0^4 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^4 = -\frac{1}{2}(e^{-16} - 1) \approx 0.500$

61. $y = 3^{3/2}$



63. $y = \log_2(x - 1)$



65. $f(x) = 3^{x-1}$

$f'(x) = 3^{x-1} \ln 3$

67. $y = x^{2x+1}$

$\ln y = (2x + 1)\ln x$

$\frac{y'}{y} = \frac{2x + 1}{x} + 2\ln x$

$y' = y\left(\frac{2x + 1}{x} + 2\ln x\right) = x^{2x+1}\left(\frac{2x + 1}{x} + 2\ln x\right)$

69. $g(x) = \log_3\sqrt{1 - x} = \frac{1}{2}\log_3(1 - x)$

$g'(x) = \frac{1}{2}\frac{-1}{(1 - x)\ln 3} = \frac{1}{2(x - 1)\ln 3}$

71. $\int (x + 1)5^{(x+1)^2}\,dx = \frac{1}{2}\frac{1}{\ln 5}5^{(x+1)^2} + C$

73. (a) $y = x^a$

$y' = ax^{a-1}$

(b) $y = a^x$

$y' = (\ln a)a^x$

(c) $y = x^x$

$\ln y = x\ln x$

$\frac{1}{y}y' = x \cdot \frac{1}{x} + (1)\ln x$

$y' = y(1 + \ln x)$

$y' = x^x(1 + \ln x)$

(d) $y = a^a$

$y' = 0$

75. $10{,}000 = Pe^{(0.07)(15)}$

$P = \frac{10{,}000}{e^{1.05}} \approx \$3499.38$

77. $P(h) = 30e^{kh}$

$P(18{,}000) = 30e^{18{,}000k} = 15$

$k = \frac{\ln(1/2)}{18{,}000} = \frac{-\ln 2}{18{,}000}$

$P(h) = 30e^{-(h\ln 2)/18{,}000}$

$P(35{,}000) = 30e^{-(35{,}000\ln 2)/18{,}000} \approx 7.79$ inches

79. $P = Ce^{0.015t}$

$2C = Ce^{0.015t}$

$2 = e^{0.015t}$

$\ln 2 = 0.015t$

$t = \frac{\ln 2}{0.015} \approx 46.21$ years

81. $\frac{dy}{dx} = \frac{x^2 + 3}{x}$

$\int dy = \int\left(x + \frac{3}{x}\right)dx$

$y = \frac{x^2}{2} + 3\ln|x| + C$

83. $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{1}{y}\,dy = \int 2x\,dx$$

$$\ln|y| = x^2 + C_1$$

$$e^{x^2 + C_1} = y$$

$$y = Ce^{x^2}$$

85. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ (homogeneous differential equation)

$$(x^2 + y^2)\,dx - 2xy\,dy = 0$$

Let $y = vx$, $dy = x\,dv + v\,dx$.

$$(x^2 + v^2x^2)\,dx - 2x(vx)(x\,dv + v\,dx) = 0$$

$$(x^2 + v^2x^2 - 2x^2v^2)\,dx - 2x^3v\,dv = 0$$

$$(x^2 - x^2v^2)\,dx = 2x^3v\,dv$$

$$(1 - v^2)\,dx = 2x\;dv$$

$$\int \frac{dx}{x} = \int \frac{2v}{1 - v^2}\,dv$$

$$\ln|x| = -\ln|1 - v^2| + C_1 = -\ln|1 - v^2| + \ln C$$

$$x = \frac{C}{1 - v^2} = \frac{C}{1 - (y/x)^2} = \frac{Cx^2}{x^2 - y^2}$$

$$1 = \frac{Cx}{x^2 - y^2} \quad \text{or} \quad C_1 = \frac{x}{x^2 - y^2}$$

87. $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2$

$y'' = 6C_2x$

$$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x(C_1 + 3C_2x^2) + (C_1x = C_2x^3)$$

$$= 6C_2x^3 - 3C_1x - 9C_2x^3 + 3C_1x + 3C_2x^3 = 0$$

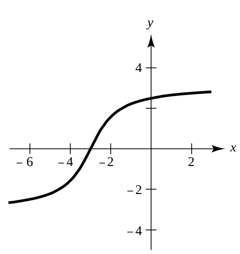
$x = 2, y = 0$: $0 = 2C_1 + 8C_2 \qquad C_1 = -4C_2$

$x = 2, y' = 4$: $4 = C_1 + 12C_2$

$$4 = (-4C_2) + 12C_2 = 8C_2 \qquad C_2 = \frac{1}{2}, C_1 = -2$$

$$y = -2x + \frac{1}{2}x^3$$

89. $f(x) = 2\arctan(x + 3)$

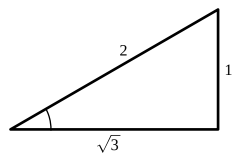


91. (a)

$$\text{Let } \theta = \arcsin \frac{1}{2}$$

$$\sin \theta = \frac{1}{2}$$

$$\sin\left(\arcsin \frac{1}{2}\right) = \sin \theta = \frac{1}{2}.$$



(b)

$$\text{Let } \theta = \arcsin \frac{1}{2}$$

$$\sin \theta = \frac{1}{2}$$

$$\cos\left(\arcsin \frac{1}{2}\right) = \cos \theta = \frac{\sqrt{3}}{2}.$$

93. $y = \tan(\arcsin x) = \dfrac{x}{\sqrt{1 - x^2}}$

$$y' = \frac{(1 - x^2)^{1/2} + x^2(1 - x^2)^{-1/2}}{1 - x^2} = (1 - x^2)^{-3/2}$$

95. $y = x \operatorname{arcsec} x$

$$y' = \frac{x}{|x|\sqrt{x^2 - 1}} + \operatorname{arcsec} x$$

97. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2} \arcsin x$

$$y' = \frac{2x \arcsin x}{\sqrt{1 - x^2}} + (\arcsin x)^2 - 2 + \frac{2\sqrt{1 - x^2}}{\sqrt{1 - x^2}} - \frac{2x}{\sqrt{1 - x^2}} \arcsin x = (\arcsin x)^2$$

99. Let $u = e^{2x}$, $du = 2e^{2x}\, dx$.

$$\int \frac{1}{e^{2x} + e^{-2x}}\, dx = \int \frac{e^{2x}}{1 + e^{4x}}\, dx = \frac{1}{2}\int \frac{1}{1 + (e^{2x})^2}(2e^{2x})\, dx = \frac{1}{2} \arctan(e^{2x}) + C$$

101. Let $u = x^2$, $du = 2x\, dx$.

$$\int \frac{x}{\sqrt{1 - x^4}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{1 - (x^2)^2}}(2x)\, dx = \frac{1}{2} \arcsin x^2 + C$$

103. Let $u = 16 + x^2$, $du = 2x\, dx$.

$$\int \frac{x}{16 + x^2}\, dx = \frac{1}{2}\int \frac{1}{16 + x^2}(2x)\, dx = \frac{1}{2} \ln(16 + x^2) + C$$

105. Let $u = \arctan\left(\dfrac{x}{2}\right)$, $du = \dfrac{2}{4 + x^2}\, dx$.

$$\int \frac{\arctan(x/2)}{4 + x^2}\, dx = \frac{1}{2}\int \left(\arctan \frac{x}{2}\right)\left(\frac{2}{4 + x^2}\right) dx = \frac{1}{4}\left(\arctan \frac{x}{2}\right)^2 + C$$

107. $\displaystyle\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}}\, dt$

$$\arcsin\left(\frac{y}{A}\right) = \sqrt{\frac{k}{m}}\, t + C$$

Since $y = 0$ when $t = 0$, you have $C = 0$. Thus,

$$\sin\left(\sqrt{\frac{k}{m}}\, t\right) = \frac{y}{A}$$

$$y = A \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

109. $y = 2x - \cosh\sqrt{x}$

$$y' = 2 - \frac{1}{2\sqrt{x}}(\sinh \sqrt{x}) = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$$

111. Let $u = x^2$, $du = 2x\, dx$.

$$\int \frac{x}{\sqrt{x^4 - 1}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{(x^2)^2 - 1}}(2x)\, dx = \frac{1}{2} \ln\left(x^2 + \sqrt{x^4 - 1}\right) + C$$