CHAPTER 2
Differentiation

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CHAPTER 2
Differentiation

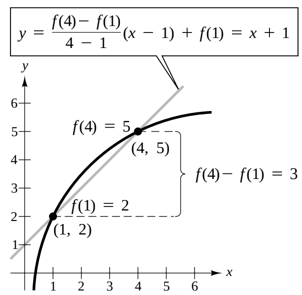
Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Odd-Numbered Exercises

1. (a) $m = 0$

(b) $m = -3$

3. (a), (b)



(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

$= \dfrac{3}{3}(x - 1) + 2$

$= 1(x - 1) + 2$

$= x + 1$

5. $f(x) = 3 - 2x$ is a line. Slope $= -2$

7. Slope at $(1, -3) = \lim_{\Delta x \to 0} \dfrac{g(1 + \Delta x) - g(1)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{(1 + \Delta x)^2 - 4 - (-3)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{1 + 2(\Delta x) + (\Delta x)^2 - 1}{\Delta x}$

$= \lim_{\Delta x \to 0} [2 + 2(\Delta x)] = 2$

9. Slope at $(0, 0) = \lim_{\Delta t \to 0} \dfrac{f(0 + \Delta t) - f(0)}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t}$

$= \lim_{\Delta t \to 0} (3 - \Delta t) = 3$

11. $f(x) = 3$

$f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{3 - 3}{\Delta x}$

$= \lim_{\Delta x \to 0} 0 = 0$

13. $f(x) = -5x$

$f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{-5(x + \Delta x) - (-5x)}{\Delta x}$

$= \lim_{\Delta x \to 0} -5 = -5$

15. $h(s) = 3 + \dfrac{2}{3}s$

$h'(s) = \lim_{\Delta s \to 0} \dfrac{h(s + \Delta s) - h(s)}{\Delta s}$

$= \lim_{\Delta s \to 0} \dfrac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s}$

$= \lim_{\Delta s \to 0} \dfrac{\frac{2}{3}\Delta s}{\Delta s} = \dfrac{2}{3}$

17. $f(x) = 2x^2 + x - 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 + (x + \Delta x) - 1] - [2x^2 + x - 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x^2 + 4x\Delta x + 2(\Delta x)^2 + x + \Delta x - 1) - (2x^2 + x - 1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x + 1) = 4x + 1$$

19. $f(x) = x^3 - 12x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^3 - 12(x + \Delta x)] - [x^3 - 12x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\Delta x - x^3 + 12x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2 - 12) = 3x^2 - 12$$

21. $f(x) = \dfrac{1}{x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x - 1} - \dfrac{1}{x - 1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x - 1) - (x + \Delta x - 1)}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x + \Delta x - 1)(x - 1)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x - 1)(x - 1)}$$

$$= -\frac{1}{(x - 1)^2}$$

23. $f(x) = \sqrt{x + 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + 1} - \sqrt{x + 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x + 1) - (x + 1)}{\Delta x\left[\sqrt{x + \Delta x + 1} + \sqrt{x + 1}\right]}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + 1} + \sqrt{x + 1}}$$

$$= \frac{1}{\sqrt{x + 1} + \sqrt{x + 1}} = \frac{1}{2\sqrt{x + 1}}$$

25. (a) $f(x) = x^2 + 1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 1] - [x^2 + 1]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

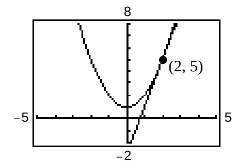
At $(2, 5)$, the slope of the tangent line is $m = 2(2) = 4$. The equation of the tangent line is

$$y - 5 = 4(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3.$$

(b)



27. (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

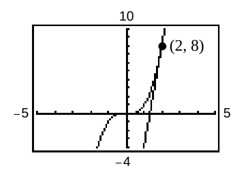
$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$. The equation of the tangent line is

$$y - 8 = 12(x - 2)$$

$$y = 12x - 16.$$

(b)



29. (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At $(1, 1)$, the slope of the tangent line is

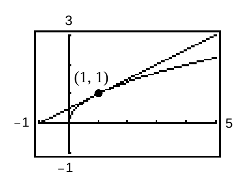
$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)



31. (a) $f(x) = \dfrac{4}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) + \dfrac{4}{x + \Delta x} - \left(x + \dfrac{4}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)}$$

$$= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $(4, 5)$, the slope of the tangent line is

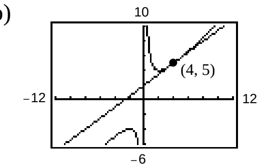
$$m = 1 - \frac{4}{16} = \frac{3}{4}$$

The equation of the tangent line is

$$y - 5 = \frac{3}{4}(x - 4)$$

$$y = \frac{3}{4}x + 2$$

(b)



33. From Exercise 27 we know that $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$. These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \qquad\qquad y = 3x + 2.$$

35. Using the limit definition of derivative,

$$f'(x) = \frac{-1}{2x\sqrt{x}}.$$

Since the slope of the given line is $-\frac{1}{2}$, we have

$$-\frac{1}{2x\sqrt{x}} = -\frac{1}{2}$$

$$x = 1.$$

Therefore, at the point $(1, 1)$ the tangent line is parallel to $x + 2y - 6 = 0$. The equation of this line is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$$y - 1 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

37. $g(5) = 2$ because the tangent line passes through $(5, 2)$

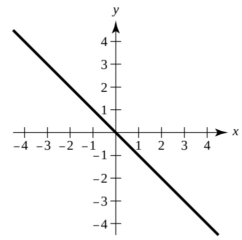
$$g'(5) = \frac{2 - 0}{5 - 9} = \frac{2}{-4} = -\frac{1}{2}$$

39. $f(x) = x \qquad f'(x) = 1 \qquad$ (b)

41. $f(x) = \sqrt{x} \qquad f'(x)$ matches (a)

(decreasing slope as $x \quad \infty$)

43.



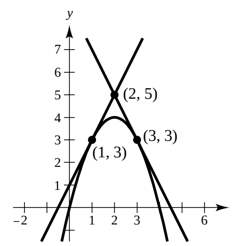
Answers will vary.

Sample answer: $y = -x$

45. (a) If $f'(c) = 3$ and f is odd, then $f'(-c) = f'(c) = 3$

(b) If $f'(c) = 3$ and f is even, then $f'(-c) = -f(c) = -3$

47. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0:



$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$

$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$

$$5 - (4x_0 - x_0^2) = 8 - 8x_0 + 2x_0^2$$

$$0 = x_0^2 - 4x_0 + 3$$

$$0 = (x_0 - 1)(x_0 - 3) \qquad x_0 = 1, 3$$

Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2. The equations of the tangent lines are

$$y - 5 = 2(x - 2) \qquad y - 5 = -2(x - 2)$$

$$y = 2x + 1 \qquad y = -2x + 9$$

49. (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.

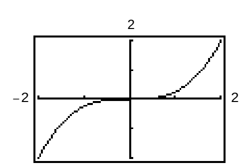
(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

51. $f(x) = \frac{1}{4}x^3$

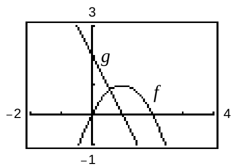
By the limit definition of the derivative we have $f'(x) = \frac{3}{4}x^2$.



x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-2	$-\frac{27}{32}$	$-\frac{1}{4}$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{27}{32}$	2
$f'(x)$	3	$\frac{27}{16}$	$\frac{3}{4}$	$\frac{3}{16}$	0	$\frac{3}{16}$	$\frac{3}{4}$	$\frac{27}{16}$	3

53. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= (2(x + 0.01) - (x + 0.01)^2 - 2x + x^2) + 100$

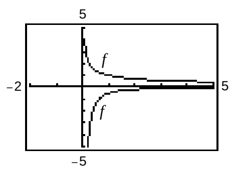


The graph of $g(x)$ is approximately the graph of $f'(x)$.

55. $f(2) = 2(4 - 2) = 4,\ f(2.1) = 2.1(4 - 2.1) = 3.99$

$f'(2) \approx \dfrac{3.99 - 4}{2.1 - 2} = -0.1$ [Exact: $f'(2) = 0$]

57. $f(x) = \dfrac{1}{\sqrt{x}}$ and $f'(x) = \dfrac{-1}{2x^{3/2}}$.



As $x \to \infty$, f is nearly horizontal and thus $f' \approx 0$.

59. $f(x) = 4 - (x - 3)^2$

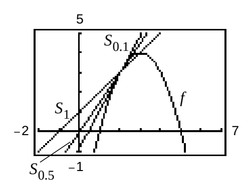
$S_{\Delta x}(x) = \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2)$

$= \dfrac{4 - (2 + \Delta x - 3)^2 - 3}{\Delta x}(x - 2) + 3 = \dfrac{1 - (\Delta x - 1)^2}{\Delta x}(x - 2) + 3 = (-\Delta x + 2)(x - 2) + 3$

(a) $\Delta x = 1$: $S_{\Delta x} = (x - 2) + 3 = x + 1$

$\Delta x = 0.5$: $S_{\Delta x} = \left(\dfrac{3}{2}\right)(x - 2) + 3 = \dfrac{3}{2}x$

$\Delta x = 0.1$: $S_{\Delta x} = \left(\dfrac{19}{10}\right)(x - 2) + 3 = \dfrac{19}{10}x - \dfrac{4}{5}$



(b) As $\Delta x \to 0$, the line approaches the tangent line to f at $(2, 3)$.

61. $f(x) = x^2 - 1,\ c = 2$

$f'(2) = \lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \dfrac{(x^2 - 1) - 3}{x - 2} = \lim_{x \to 2} \dfrac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$

63. $f(x) = x^3 + 2x^2 + 1,\ c = -2$

$f'(-2) = \lim_{x \to -2} \dfrac{f(x) - f(-2)}{x + 2} = \lim_{x \to -2} \dfrac{x^2(x + 2)}{x + 2} = \lim_{x \to -2} \dfrac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x \to -2} x^2 = 4$

65. $g(x) = \sqrt{|x|},\ c = 0$

$g'(0) = \lim_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \dfrac{\sqrt{|x|}}{x}$. Does not exist.

As $x \to 0^-$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{-1}{\sqrt{x}} \to -\infty$

As $x \to 0^+$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{1}{\sqrt{x}} \to \infty$

67. $f(x) = (x - 6)^{2/3},\ c = 6$

$f'(6) = \lim_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$

$= \lim_{x \to 6} \dfrac{(x - 6)^{2/3} - 0}{x - 6}$

$= \lim_{x \to 6} \dfrac{1}{(x - 6)^{1/3}}$

Does not exist.

69. $h(x) = (x + 5), c = -5$

$$h'(-5) = \lim_{x \to -5} \frac{h(x) - h(-5)}{x - (-5)}$$
$$= \lim_{x \to -5} \frac{|x + 5| - 0}{x + 5}$$
$$= \lim_{x \to -5} \frac{|x + 5|}{x + 5}$$

Does not exist.

71. $f(x)$ is differentiable everywhere except at $x = -3$. (Sharp turn in the graph.)

73. $f(x)$ is differentiable everywhere except at $x = -1$. (Discontinuity)

75. $f(x)$ is differentiable everywhere except at $x = 3$. (Sharp turn in the graph)

77. $f(x)$ is differentiable on the interval $(1, \infty)$. (At $x = 1$ the tangent line is vertical)

79. $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

81. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

83. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x - 1)^3 - 0}{x - 1}$$
$$= \lim_{x \to 1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x - 1)^2 - 0}{x - 1}$$
$$= \lim_{x \to 1^+} (x - 1) = 0.$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. $(f'(1) = 0)$

85. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

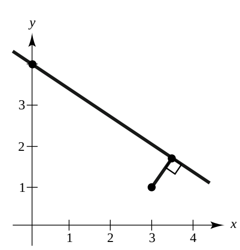
The derivative from the left is $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(x^2 + 1) - 5}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4.$

The derivative from the right is $\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x \to 2^+} 4 = 4.$

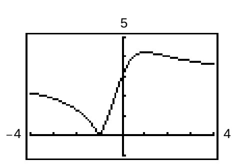
The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. $(f'(2) = 4)$

87. (a) The distance from $(3, 1)$ to the line $mx - y + 4 = 0$ is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$
$$= \frac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}} = \frac{|3m + 3|}{\sqrt{m^2 + 1}}.$$



(b)



The function d is not differentiable at $m = -1$. This corresponds to the line $y = -x + 4$, which passes through the point $(3, 1)$.

89. False. the slope is $\lim_{\triangle x \to 0} \dfrac{f(2 + \triangle x) - f(2)}{\triangle x}$.

91. False. If the derivative from the left of a point does not equal the derivative from the right of a point, then the derivative does not exist at that point. For example, if $f(x) = |x|$, then the derivative from the left at $x = 0$ is -1 and the derivative from the right at $x = 0$ is 1. At $x = 0$, the derivative does not exist.

93. $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, we have $-|x| \le x \sin(1/x) \le |x|,\ x \neq 0$. Thus, $\lim_{x \to 0} x \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \left(\sin \frac{1}{x}\right).$$

Since this limit does not exist (it oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again we have $-x^2 \le x^2 \sin(1/x) \le x^2, x \neq 0$. Thus, $\lim_{x \to 0} x^2 \sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$. Using the alternative form of the derivative again we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

Section 2.2 Basic Differentiation Rules and Rates of Change

1. (a) $y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2}$

$y'(1) = \frac{1}{2}$

(b) $y = x^{3/2}$

$y' = \frac{3}{2}x^{1/2}$

$y'(1) = \frac{3}{2}$

(c) $y = x^2$

$y' = 2x$

$y'(1) = 2$

(d) $y = x^3$

$y' = 3x^2$

$y'(1) = 3$

3. $y = 8$

$y' = 0$

5. $y = x^6$

$y' = 6x^5$

7. $y = \dfrac{1}{x^7} = x^{-7}$

$y' = -7x^{-8} = \dfrac{-7}{x^8}$

9. $y = \sqrt[5]{x} = x^{1/5}$

$y' = \dfrac{1}{5}x^{-4/5} = \dfrac{1}{5x^{4/5}}$

11. $f(x) = x + 1$

$f'(x) = 1$

13. $f(t) = -2t^2 + 3t - 6$

$f'(x) = -4t + 3$

15. $g(x) = x^2 + 4x^3$

$g'(x) = 2x + 12x^2$

17. $s(t) = t^3 - 2t + 4$

$s'(t) = 3t^2 - 2$

19. $y = \dfrac{\pi}{2} \sin \theta - \cos \theta$

$y' = \dfrac{\pi}{2} \cos \theta + \sin \theta$

21. $y = x^2 - \dfrac{1}{2} \cos x$

$y' = 2x + \dfrac{1}{2} \sin x$

23. $y = \dfrac{1}{x} - 3 \sin x$

$y' = -\dfrac{1}{x^2} - 3 \cos x$

Function	Rewrite	Derivative	Simplify
25. $y = \dfrac{5}{2x^2}$	$y = \dfrac{5}{2}x^{-2}$	$y' = -5x^{-3}$	$y' = \dfrac{-5}{x^3}$
27. $y = \dfrac{3}{(2x)^3}$	$y = \dfrac{3}{8}x^{-3}$	$y' = \dfrac{-9}{8}x^{-4}$	$y' = \dfrac{-9}{8x^4}$
29. $y = \dfrac{\sqrt{x}}{x}$	$y = x^{-1/2}$	$y' = -\dfrac{1}{2}x^{-3/2}$	$y' = -\dfrac{1}{2x^{3/2}}$

31. $f(x) = \dfrac{3}{x^2} = 3x^{-2}$, $(1, 3)$

$f'(x) = -6x^{-3} = \dfrac{-6}{x^3}$

$f'(1) = -6$

33. $f(x) = -\dfrac{1}{2} + \dfrac{7}{5}x^3$, $\left(0, -\dfrac{1}{2}\right)$

$f'(x) = \dfrac{21}{5}x^2$

$f'(0) = 0$

35. $y = (2x + 1)^2$, $(0, 1)$

$= 4x^2 + 4x + 1$

$y' = 8x + 4$

$y'(0) = 4$

37. $f(\theta) = 4\sin\theta - \theta$, $(0, 0)$

$f'(\theta) = 4\cos\theta - 1$

$f'(0) = 4(1) - 1 = 3$

39. $f(x) = x^2 + 5 - 3x^{-2}$

$f'(x) = 2x + 6x^{-3} = 2x + \dfrac{6}{x^3}$

41. $g(t) = t^2 - \dfrac{4}{t^3} = t^2 - 4t^{-3}$

$g'(t) = 2t + 12t^{-4} = 2t + \dfrac{12}{t^4}$

43. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2} = x - 3 + 4x^{-2}$

$f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3}$

45. $y = x(x^2 + 1) = x^3 + x$

$y' = 3x^2 + 1$

47. $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$f'(x) = \dfrac{1}{2}x^{-1/2} + 2x^{-2/3} = \dfrac{1}{2\sqrt{x}} + \dfrac{2}{x^{2/3}}$

49. $h(s) = s^{4/5} - s^{2/3}$

$h'(s) = \dfrac{4}{5}s^{-4/5} - \dfrac{2}{3}s^{-1/3} = \dfrac{4}{5s^{1/5}} - \dfrac{2}{3s^{1/3}}$

51. $f(x) = 6\sqrt{x} + 5\cos x = 6x^{1/2} + 5\cos x$

$f'(x) = 3x^{-1/2} - 5\sin x = \dfrac{3}{\sqrt{x}} - 5\sin x$

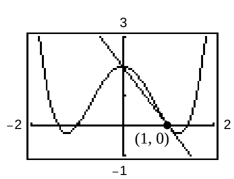
53. (a) $y = x^4 - 3x^2 + 2$

$y' = 4x^3 - 6x$

At $(1, 0)$: $y' = 4(1)^3 - 6(1) = -2$.

Tangent line: $y - 0 = -2(x - 1)$

$2x + y - 2 = 0$

(b) 

55. (a) $f(x) = \dfrac{2}{\sqrt[4]{x^3}} = 2x^{-3/4}$

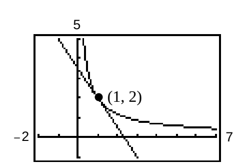
$f'(x) = \dfrac{-3}{2}x^{-7/4} = \dfrac{-3}{2x^{7/4}}$

At $(1, 2)$, $f'(1) = \dfrac{-3}{2}$

Tangent line: $y - 2 = -\dfrac{3}{2}(x - 1)$

$y = -\dfrac{3}{2}x + \dfrac{7}{2}$

$3x + 2y - 7 = 0$

(b) 

57. $y = x^4 - 8x^2 + 2$

$y' = 4x^3 - 16x$

$= 4x(x^2 - 4)$

$= 4x(x - 2)(x + 2)$

$y' = 0 \quad x = 0, \pm 2$

Horizontal tangents: $(0, 2), (2, -14), (-2, -14)$

59. $y = \dfrac{1}{x^2} = x^{-2}$

$y' = -2x^{-3} = \dfrac{-2}{x^3}$ cannot equal zero.

Therefore, there are no horizontal tangents.

61. $y = x + \sin x, 0 \quad x < 2\pi$

$y' = 1 + \cos x = 0$

$\cos x = -1 \quad x = \pi$

At $x = \pi, y = \pi$.

Horizontal tangent: (π, π)

63. $x^2 - kx = 4x - 9$ Equate functions

$2x - k = 4$ Equate derivatives

Hence, $k = 2x - 4$ and

$x^2 - (2x - 4)x = 4x - 9 \quad -x^2 = -9 \quad x = \pm 3.$

For $x = 3, k = 2$ and for $x = -3, k = -10$.

65. $\dfrac{k}{x} = -\dfrac{3}{4}x + 3$ Equate functions

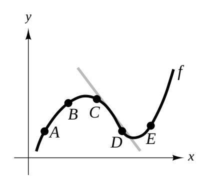
$-\dfrac{k}{x^2} = -\dfrac{3}{4}$ Equate derivatives

Hence, $k = \dfrac{3}{4}x^2$ and $\dfrac{\frac{3}{4}x^2}{x} = \dfrac{-3}{4}x + 3 \quad \dfrac{3}{4}x = -\dfrac{3}{4}x + 3 \quad \dfrac{3}{2}x = 3 \quad x = 2 \quad k = 3.$

67. (a) The slope appears to be steepest between A and B.

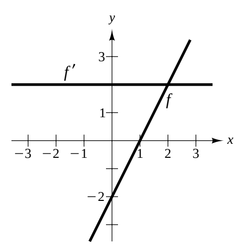
(b) The average rate of change between A and B is **greater** than the instantaneous rate of change at B.

(c)



69. $g(x) = f(x) + 6 \quad g'(x) = f'(x)$

71.



If f is linear then its derivative is a constant function.

$f(x) = ax + b$

$f'(x) = a$

73. Let (x_1, y_1) and (x_2, y_2) be the points of tangency on $y = x^2$ and $y = -x^2 + 6x - 5$, respectively. The derivatives of these functions are

$$y' = 2x \qquad m = 2x_1 \quad \text{and} \quad y' = -2x + 6 \qquad m = -2x_2 + 6.$$

$$m = 2x_1 = -2x_2 + 6$$

$$x_1 = -x_2 + 3$$

Since $y_1 = x_1^2$ and $y_2 = -x_2^2 + 6x_2 - 5$,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-x_2^2 + 6x_2 - 5) - (-x_1^2)}{x_2 - x_1} = -2x_2 + 6.$$

$$\frac{(-x_2^2 + 6x_2 - 5) - (-x_2 + 3)^2}{x_2 - (-x_2 + 3)} = -2x_2 + 6$$

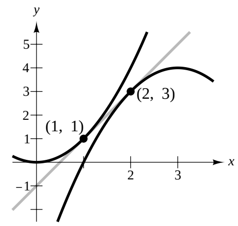
$$(-x_2^2 + 6x_2 - 5) - (x_2^2 - 6x_2 + 9) = (-2x_2 + 6)(2x_2 - 3)$$

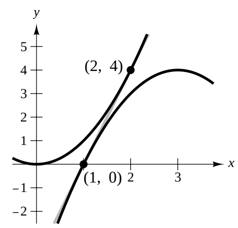
$$-2x_2^2 + 12x_2 - 14 = -4x_2^2 + 18x_2 - 18$$

$$2x_2^2 - 6x_2 + 4 = 0$$

$$2(x_2 - 2)(x_2 - 1) = 0$$

$$x_2 = 1 \text{ or } 2$$





$x_2 = 1 \qquad y_2 = 0, x_1 = 2$ and $y_1 = 4$

Thus, the tangent line through $(1, 0)$ and $(2, 4)$ is

$$y - 0 = \left(\frac{4 - 0}{2 - 1}\right)(x - 1) \qquad y = 4x - 4.$$

$x_2 = 2 \qquad y_2 = 3, x_1 = 1$ and $y_1 = 1$

Thus, the tangent line through $(2, 3)$ and $(1, 1)$ is

$$y - 1 = \left(\frac{3 - 1}{2 - 1}\right)(x - 1) \qquad y = 2x - 1.$$

75. $f(x) = \sqrt{x}, \ (-4, 0)$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{x}} = \frac{0 - y}{-4 - x}$$

$$4 + x = 2\sqrt{x}y$$

$$4 + x = 2\sqrt{x}\sqrt{x}$$

$$4 + x = 2x$$

$$x = 4, y = 2$$

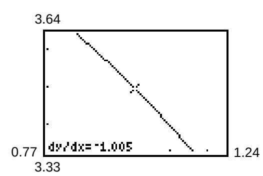
The point $(4, 2)$ is on the graph of f.

Tangent line: $\quad y - 2 = \dfrac{0 - 2}{-4 - 4}(x - 4)$

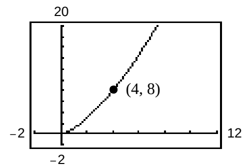
$$4y - 8 = x - 4$$

$$0 = x - 4y + 4$$

77. $f'(1) = -1$



79. (a) One possible secant is between (3.9, 7.7019) and (4, 8):



$$y - 8 = \frac{8 - 7.7019}{4 - 3.9}(x - 4)$$

$$y - 8 = 2.981(x - 4)$$

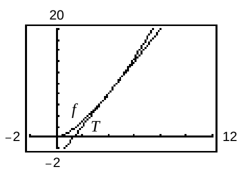
$$y = S(x) = 2.981x - 3.924$$

(b) $f'(x) = \frac{3}{2}x^{1/2}$ $\quad f'(4) = \frac{3}{2}(2) = 3$

$T(x) = 3(x - 4) + 8 = 3x - 4$

$S(x)$ is an approximation of the tangent line $T(x)$.

(c) As you move further away from (4, 8), the accuracy of the approximation T gets worse.



(d)

Δx	−3	−2	−1	−0.5	−0.1	0	0.1	0.5	1	2	3
$f(4 + \Delta x)$	1	2.828	5.196	6.548	7.702	8	8.302	9.546	11.180	14.697	18.520
$T(4 + \Delta x)$	−1	2	5	6.5	7.7	8	8.3	9.5	11	14	17

81. False. Let $f(x) = x^2$ and $g(x) = x^2 + 4$. Then $f'(x) = g'(x) = 2x$, but $f(x) \neq g(x)$.

83. False. If $y = \pi^2$, then $dy/dx = 0$. (π^2 is a constant.)

85. True. If $g(x) = 3f(x)$, then $g'(x) = 3f'(x)$.

87. $f(t) = 2t + 7, [1, 2]$

$f'(t) = 2$

Instantaneous rate of change is the constant 2.
Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{[2(2) + 7] - [2(1) + 7]}{1} = 2$$

(These are the same because f is a line of slope 2.)

89. $f(x) = -\frac{1}{x}, [1, 2]$

$$f'(x) = \frac{1}{x^2}$$

Instantaneous rate of change:

$(1, -1)$ $\quad f'(1) = 1$

$\left(2, -\frac{1}{2}\right)$ $\quad f'(2) = \frac{1}{4}$

Average rate of change:

$$\frac{f(2) - f(1)}{2 - 1} = \frac{(-1/2) - (-1)}{2 - 1} = \frac{1}{2}$$

91. (a) $s(t) = -16t^2 + 1362$

$v(t) = -32t$

(b) $\dfrac{s(2) - s(1)}{2 - 1} = 1298 - 1346 = -48$ ft/sec

(c) $v(t) = s'(t) = -32t$

When $t = 1$: $v(1) = -32$ ft/sec.

When $t = 2$: $v(2) = -64$ ft/sec.

(d) $-16t^2 + 1362 = 0$

$$t^2 = \frac{1362}{16} \qquad t = \frac{\sqrt{1362}}{4} \approx 9.226 \text{ sec}$$

(e) $v\left(\dfrac{\sqrt{1362}}{4}\right) = -32\left(\dfrac{\sqrt{1362}}{4}\right)$

$= -8\sqrt{1362} \approx -295.242$ ft/sec

93. $s(t) = -4.9t^2 + v_0t + s_0$

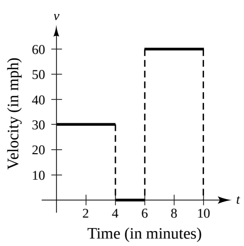
$= -4.9t^2 + 120t$

$v(t) = -9.8t + 120$

$v(5) = -9.8(5) + 120 = 71$ m/sec

$v(10) = -9.8(10) + 120 = 22$ m/sec

95.



(The velocity has been converted to miles per hour)

97. $v = 40 \text{ mph} = \frac{2}{3}$ mi/min

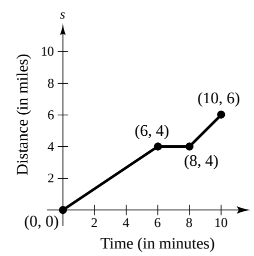
$\left(\frac{2}{3} \text{ mi/min}\right)(6 \text{ min}) = 4$ mi

$v = 0 \text{ mph} = 0$ mi/min

$(0 \text{ mi/min})(2 \text{ min}) = 0$ mi

$v = 60 \text{ mph} = 1$mi/min

$(1 \text{ mi/min})(2 \text{ min}) = 2$ mi



99. (a) Using a graphing utility, you obtain

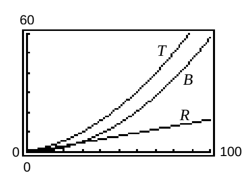
$R = 0.167v - 0.02.$

(b) Using a graphing utility, you obtain

$B = 0.00586v^2 - 0.0239v + 0.46.$

(c) $T = R + B = 0.00586v^2 + 0.1431v + 0.44$

(d)



(e) $\dfrac{dT}{dv} = 0.01172v + 0.1431$

For $v = 40$, $T'(40) \approx 0.612$.

For $v = 80$, $T'(80) \approx 1.081$.

For $v = 100$, $T'(100) \approx 1.315$.

(f) For increasing speeds, the total stopping distance increases.

101. $A = s^2, \dfrac{dA}{ds} = 2s$

When $s = 4$ m,

$\dfrac{dA}{ds} = 8$ square meters per meter change in s.

103.

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

$$\frac{dC}{dQ} = -\frac{1{,}008{,}000}{Q^2} + 6.3$$

$C(351) - C(350) \approx 5083.095 - 5085 \approx -\$1.91$

When $Q = 350$, $\dfrac{dC}{dQ} \approx -\$1.93$.

105. (a) $f'(1.47)$ is the rate of change of the amount of gasoline sold when the price is \$1.47 per gallon.

(b) $f'(1.47)$ is usually negative. As prices go up, sales go down.

107. $y = ax^2 + bx + c$

Since the parabola passes through $(0, 1)$ and $(1, 0)$, we have

$(0, 1)$: $1 = a(0)^2 + b(0) + c \qquad c = 1$

$(1, 0)$: $0 = a(1)^2 + b(1) + 1 \qquad b = -a - 1.$

Thus, $y = ax^2 + (-a - 1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point $(1, 0)$.

$$y' = 2ax + (-a - 1)$$
$$1 = 2a(1) + (-a - 1)$$
$$1 = a - 1$$
$$a = 2$$
$$b = -a - 1 = -3$$

Therefore, $y = 2x^2 - 3x + 1$.

109. $y = x^3 - 9x$

$y' = 3x^2 - 9$

Tangent lines through $(1, -9)$:

$$y + 9 = (3x^2 - 9)(x - 1)$$
$$(x^3 - 9x) + 9 = 3x^3 - 3x^2 - 9x + 9$$
$$0 = 2x^3 - 3x^2 = x^2(2x - 3)$$
$$x = 0 \text{ or } x = \tfrac{3}{2}$$

The points of tangency are $(0, 0)$ and $\left(\frac{3}{2}, -\frac{81}{8}\right)$. At $(0, 0)$ the slope is $y'(0) = -9$. At $\left(\frac{3}{2}, -\frac{81}{8}\right)$ the slope is $y'\left(\frac{3}{2}\right) = -\frac{9}{4}$.

Tangent lines:

$y - 0 = -9(x - 0)$ and $y + \frac{81}{8} = -\frac{9}{4}\left(x - \frac{3}{2}\right)$

$y = -9x$ $\qquad y = -\frac{9}{4}x - \frac{27}{4}$

$9x + y = 0$ $\qquad 9x + 4y + 27 = 0$

111. $f(x) = \begin{cases} ax^3, & x \quad 2 \\ x^2 + b, & x > 2 \end{cases}$

f must be continuous at $x = 2$ to be differentiable at $x = 2$.

$$\left.\begin{aligned} \lim_{x \quad 2^-} f(x) &= \lim_{x \quad 2^-} ax^3 = 8a \\ \lim_{x \quad 2^+} f(x) &= \lim_{x \quad 2^+} (x^2 + b) = 4 + b \end{aligned}\right\} \quad \begin{aligned} 8a &= 4 + b \\ 8a - 4 &= b \end{aligned}$$

$$f'(x) = \begin{cases} 3ax^2, & x < 2 \\ 2x, & x > 2 \end{cases}$$

For f to be differentiable at $x = 2$, the left derivative must equal the right derivative.

$$3a(2)^2 = 2(2)$$
$$12a = 4$$
$$a = \tfrac{1}{3}$$
$$b = 8a - 4 = -\tfrac{4}{3}$$

113. Let $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cos x(\cos \Delta x - 1)}{\Delta x} - \lim_{\Delta x \to 0} \sin x\left(\frac{\sin \Delta x}{\Delta x}\right) \\ &= 0 - \sin x(1) = -\sin x \end{aligned}$$

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

1. $g(x) = (x^2 + 1)(x^2 - 2x)$

$$\begin{aligned} g'(x) &= (x^2 + 1)(2x - 2) + (x^2 - 2x)(2x) \\ &= 2x^3 - 2x^2 + 2x - 2 + 2x^3 - 4x^2 \\ &= 4x^3 - 6x^2 + 2x - 2 \end{aligned}$$

3. $h(t) = \sqrt[3]{t}(t^2 + 4) = t^{1/3}(t^2 + 4)$

$$\begin{aligned} h'(t) &= t^{1/3}(2t) + (t^2 + 4)\frac{1}{3}t^{-2/3} \\ &= 2t^{4/3} + \frac{t^2 + 4}{3t^{2/3}} \\ &= \frac{7t^2 + 4}{3t^{2/3}} \end{aligned}$$

5. $f(x) = x^3 \cos x$

$$\begin{aligned} f'(x) &= x^3(-\sin x) + \cos x(3x^2) \\ &= 3x^2 \cos x - x^3 \sin x \end{aligned}$$

7. $f(x) = \dfrac{x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

9. $h(x) = \dfrac{\sqrt[3]{x}}{x^3 + 1} = \dfrac{x^{1/3}}{x^3 + 1}$

$$\begin{aligned} h'(x) &= \frac{(x^3 + 1)\frac{1}{3}x^{-2/3} - x^{1/3}(3x^2)}{(x^3 + 1)^2} \\ &= \frac{(x^3 + 1) - x(9x^2)}{3x^{2/3}(x^3 + 1)^2} \\ &= \frac{1 - 8x^3}{3x^{2/3}(x^3 + 1)^2} \end{aligned}$$

11. $g(x) = \dfrac{\sin x}{x^2}$

$$g'(x) = \frac{x^2(\cos x) - \sin x(2x)}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

13. $f(x) = (x^3 - 3x)(2x^2 + 3x + 5)$

$$\begin{aligned} f'(x) &= (x^3 - 3x)(4x + 3) + (2x^2 + 3x + 5)(3x^2 - 3) \\ &= 10x^4 + 12x^3 - 3x^2 - 18x - 15 \end{aligned}$$

$f'(0) = -15$

15. $f(x) = \dfrac{x^2 - 4}{x - 3}$

$$\begin{aligned} f'(x) &= \frac{(x - 3)(2x) - (x^2 - 4)(1)}{(x - 3)^2} = \frac{2x^2 - 6x - x^2 + 4}{(x - 3)^2} \\ &= \frac{x^2 - 6x + 4}{(x - 3)^2} \end{aligned}$$

$$f'(1) = \frac{1 - 6 + 4}{(1 - 3)^2} = -\frac{1}{4}$$

17. $f(x) = x \cos x$

$$f'(x) = (x)(-\sin x) + (\cos x)(1) = \cos x - x \sin x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{8}(4 - \pi)$$

Function	Rewrite	Derivative	Simplify
19. $y = \dfrac{x^2 + 2x}{3}$	$y = \dfrac{1}{3}x^2 + \dfrac{2}{3}x$	$y' = \dfrac{2}{3}x + \dfrac{2}{3}$	$y' = \dfrac{2x + 2}{3}$
21. $y = \dfrac{7}{3x^3}$	$y = \dfrac{7}{3}x^{-3}$	$y' = -7x^{-4}$	$y' = -\dfrac{7}{x^4}$
23. $y = \dfrac{4x^{3/2}}{x}$	$y = 4\sqrt{x}, x > 0$	$y' = 2x^{-1/2}$	$y' = \dfrac{2}{\sqrt{x}}$

25. $f(x) = \dfrac{3 - 2x - x^2}{x^2 - 1}$

$$f'(x) = \frac{(x^2 - 1)(-2 - 2x) - (3 - 2x - x^2)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^2 - 4x + 2}{(x^2 - 1)^2} = \frac{2(x - 1)^2}{(x^2 - 1)^2}$$

$$= \frac{2}{(x + 1)^2}, x \neq 1$$

27. $f(x) = x\left(1 - \dfrac{4}{x + 3}\right) = x - \dfrac{4x}{x + 3}$

$$f'(x) = 1 - \frac{(x + 3)4 - 4x(1)}{(x + 3)^2} = \frac{(x^2 + 6x + 9) - 12}{(x + 3)^2}$$

$$= \frac{x^2 + 6x - 3}{(x + 3)^2}$$

29. $f(x) = \dfrac{2x + 5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$

$$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$$

$$= \frac{2x - 5}{2x\sqrt{x}} = \frac{2x - 5}{2x^{3/2}}$$

31. $h(s) = (s^3 - 2)^2 = s^6 - 4s^3 + 4$

$h'(s) = 6s^5 - 12s^2 = 6s^2(s^3 - 2)$

33. $f(x) = \dfrac{2 - \dfrac{1}{x}}{x - 3} = \dfrac{2x - 1}{x(x - 3)} = \dfrac{2x - 1}{x^2 - 3x}$

$$f'(x) = \frac{(x^2 - 3x)2 - (2x - 1)(2x - 3)}{(x^2 - 3x)^2} = \frac{2x^2 - 6x - 4x^2 + 8x - 3}{(x^2 - 3x)^2}$$

$$= \frac{-2x^2 + 2x - 3}{(x^2 - 3x)^2} = -\frac{2x^2 - 2x + 3}{x^2(x - 3)^2}$$

35. $f(x) = (3x^3 + 4x)(x - 5)(x + 1)$

$$f'(x) = (9x^2 + 4)(x - 5)(x + 1) + (3x^3 + 4x)(1)(x + 1) + (3x^3 + 4x)(x - 5)(1)$$

$$= (9x^2 + 4)(x^2 - 4x - 5) + 3x^4 + 3x^3 + 4x^2 + 4x + 3x^4 - 15x^3 + 4x^2 - 20x$$

$$= 9x^4 - 36x^3 - 41x^2 - 16x - 20 + 6x^4 - 12x^3 + 8x^2 - 16x$$

$$= 15x^4 - 48x^3 - 33x^2 - 32x - 20$$

37. $f(x) = \dfrac{x^2 + c^2}{x^2 - c^2}$

$$f'(x) = \frac{(x^2 - c^2)(2x) - (x^2 + c^2)(2x)}{(x^2 - c^2)^2}$$

$$= \frac{-4xc^2}{(x^2 - c^2)^2}$$

39. $f(x) = t^2 \sin t$

$$f'(t) = t^2 \cos t + 2t \sin t$$

$$= t(t \cos t + 2 \sin t)$$

41. $f(t) = \dfrac{\cos t}{t}$

$$f'(t) = \frac{-t \sin t - \cos t}{t^2} = -\frac{t \sin t + \cos t}{t^2}$$

43. $f(x) = -x + \tan x$

$$f'(x) = -1 + \sec^2 x = \tan^2 x$$

45. $g(t) = \sqrt[4]{t} + 8 \sec t = t^{1/4} + 8 \sec t$

$$g'(t) = \frac{1}{4}t^{-3/4} + 8 \sec t \tan t = \frac{1}{4t^{3/4}} + 8 \sec t \tan t$$

47. $y = \dfrac{3(1 - \sin x)}{2 \cos x} = \dfrac{3}{2}(\sec x - \tan x)$

$$y' = \frac{3}{2}(\sec x \tan x - \sec^2 x) = \frac{3}{2}\sec x(\tan x - \sec x)$$

$$= \frac{3}{2}(\sec x \tan x - \tan^2 x - 1)$$

49. $y = -\csc x - \sin x$

$$y' = \csc x \cot x - \cos x$$

$$= \frac{\cos x}{\sin^2 x} - \cos x$$

$$= \cos x(\csc^2 x - 1)$$

$$= \cos x \cot^2 x$$

51. $f(x) = x^2 \tan x$

$$f'(x) = x^2 \sec^2 x + 2x \tan x$$

$$= x(x \sec^2 x + 2 \tan x)$$

53. $y = 2x \sin x + x^2 \cos x$

$$y' = 2x \cos x + 2 \sin x + x^2(-\sin x) + 2x \cos x$$

$$= 4x \cos x + 2 \sin x - x^2 \sin x$$

55. $g(x) = \left(\dfrac{x + 1}{x + 2}\right)(2x - 5)$

$$g'(x) = \frac{2x^2 + 8x - 1}{(x + 2)^2}$$ (form of answer may vary)

57. $g(\theta) = \dfrac{\theta}{1 - \sin \theta}$

$$g'(\theta) = \frac{1 - \sin \theta + \theta \cos \theta}{(\sin \theta - 1)^2}$$ (form of answer may vary)

59. $y = \dfrac{1 + \csc x}{1 - \csc x}$

$$y' = \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2} = \frac{-2 \csc x \cot x}{(1 - \csc x)^2}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{-2(2)\left(\sqrt{3}\right)}{(1 - 2)^2} = -4\sqrt{3}$$

61. $h(t) = \dfrac{\sec t}{t}$

$$h'(t) = \frac{t(\sec t \tan t) - (\sec t)(1)}{t^2}$$

$$= \frac{\sec t(t \tan t - 1)}{t^2}$$

$$h'(\pi) = \frac{\sec \pi(\pi \tan \pi - 1)}{\pi^2} = \frac{1}{\pi^2}$$

63. (a) $f(x) = (x^3 - 3x + 1)(x + 2), \quad (1, -3)$

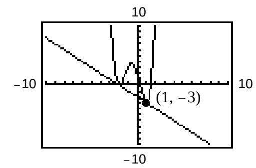
$f'(x) = (x^3 - 3x + 1)(1) + (x + 2)(3x^2 - 3)$

$= 4x^3 + 6x^2 - 6x - 5$

$f'(1) = -1 =$ slope at $(1, -3)$.

Tangent line: $y + 3 = -1(x - 1) \qquad y = -x - 2$

(b)



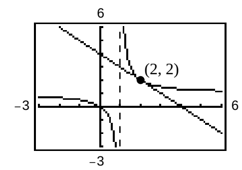
65. (a) $f(x) = \dfrac{x}{x - 1}, (2, 2)$

$f'(x) = \dfrac{(x - 1)(1) - x(1)}{(x - 1)^2} = \dfrac{-1}{(x - 1)^2}$

$f'(2) = \dfrac{-1}{(2 - 1)^2} = -1 =$ slope at $(2, 2)$.

Tangent line: $y - 2 = -1(x - 2) \qquad y = -x + 4$

(b)



67. (a) $f(x) = \tan x, \ \left(\dfrac{\pi}{4}, 1\right)$

$f'(x) = \sec^2 x$

$f'\left(\dfrac{\pi}{4}\right) = 2 =$ slope at $\left(\dfrac{\pi}{4}, 1\right)$.

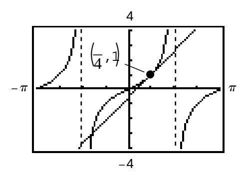
Tangent line:

$y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$

$y - 1 = 2x - \dfrac{\pi}{2}$

$4x - 2y - \pi + 2 = 0$

(b)



69. $f(x) = \dfrac{x^2}{x - 1}$

$f'(x) = \dfrac{(x - 1)(2x) - x^2(1)}{(x - 1)^2}$

$= \dfrac{x^2 - 2x}{(x - 1)^2} = \dfrac{x(x - 2)}{(x - 1)^2}$

$f'(x) = 0$ when $x = 0$ or $x = 2$.

Horizontal tangents are at $(0, 0)$ and $(2, 4)$.

71. $f'(x) = \dfrac{(x + 2)3 - 3x(1)}{(x + 2)^2} = \dfrac{6}{(x + 2)^2}$

$g'(x) = \dfrac{(x + 2)5 - (5x + 4)(1)}{(x + 2)^2} = \dfrac{6}{(x + 2)^2}$

$g(x) = \dfrac{5x + 4}{(x + 2)} = \dfrac{3x}{(x + 2)} + \dfrac{2x + 4}{(x + 2)} = f(x) + 2$

f and g differ by a constant.

73. $f(x) = x^n \sin x$

$f'(x) = x^n \cos x + nx^{n-1} \sin x$

$= x^{n-1}(x \cos x + n \sin x)$

When $n = 1$: $f'(x) = x \cos x + \sin x$.

When $n = 2$: $f'(x) = x(x \cos x + 2 \sin x)$.

When $n = 3$: $f'(x) = x^2(x \cos x + 3 \sin x)$.

When $n = 4$: $f'(x) = x^3(x \cos x + 4 \sin x)$.

For general n, $f'(x) = x^{n-1}(x \cos x + n \sin x)$.

75. Area $= A(t) = (2t + 1)\sqrt{t} = 2t^{3/2} + t^{1/2}$

$A'(t) = 2\left(\dfrac{3}{2}t^{1/2}\right) + \dfrac{1}{2}t^{-1/2}$

$= 3t^{1/2} + \dfrac{1}{2}t^{-1/2}$

$= \dfrac{6t + 1}{2\sqrt{t}}$ cm²/sec

77. $C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \quad 1 \qquad x$

$$\frac{dC}{dx} = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

(a) When $x = 10$: $\frac{dC}{dx} = -\$38.13$.

(b) When $x = 15$: $\frac{dC}{dx} = -\$10.37$.

(c) When $x = 20$: $\frac{dC}{dx} = -\$3.80$.

As the order size increases, the cost per item decreases.

79. $P(t) = 500\left[1 + \frac{4t}{50 + t^2}\right]$

$$P'(t) = 500\left[\frac{(50 + t^2)(4) - (4t)(2t)}{(50 + t^2)^2}\right]$$

$$= 500\left[\frac{200 - 4t^2}{(50 + t^2)^2}\right]$$

$$= 2000\left[\frac{50 = t^2}{(50 + t^2)^2}\right]$$

$P'(2) \approx 31.55$ bacteria per hour

81. (a) $\sec x = \frac{1}{\cos x}$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos x \cos x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(b) $\csc x = \frac{1}{\sin x}$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = \frac{(\sin x)(0) - (1)(\cos x)}{(\sin x)^2} = -\frac{\cos x}{\sin x \sin x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

(c) $\cot x = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x(-\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

83. $f(x) = 4x^{3/2}$

$f'(x) = 6x^{1/2}$

$f''(x) = 3x^{-1/2} = \frac{3}{\sqrt{x}}$

85. $f(x) = \frac{x}{x - 1}$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}$$

87. $f(x) = 3 \sin x$

$f'(x) = 3 \cos x$

$f''(x) = -3 \sin x$

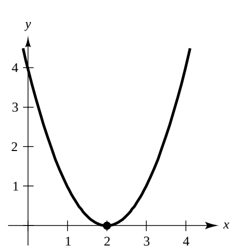
89. $f'(x) = x^2$

$f''(x) = 2x$

91. $f'''(x) = 2\sqrt{x}$

$$f^{(4)}(x) = \frac{1}{2}(2)x^{-1/2} = \frac{1}{\sqrt{x}}$$

93.



$f(2) = 0$

One such function is $f(x) = (x - 2)^2$.

95. $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4$

$= 0$

97. $f(x) = \frac{g(x)}{h(x)}$

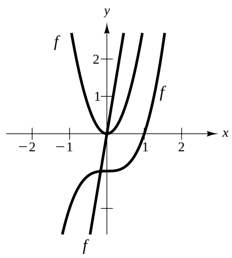
$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$

$$f'(2) = \frac{h(2)g'(2) - g(2)h'(2)}{[h(2)]^2}$$

$$= \frac{(-1)(-2) - (3)(4)}{(-1)^2}$$

$$= -10$$

99.



It appears that f is cubic; so f' would be quadratic and f'' would be linear.

101. $v(t) = 36 - t^2,\ 0 \quad t \quad 6$

$a(t) = -2t$

$v(3) = 27$ m/sec

$a(3) = -6$ m/sec

The speed of the object is decreasing.

103. $v(t) = \dfrac{100t}{2t + 15}$

$$a(t) = \frac{(2t + 15)(100) - (100t)(2)}{(2t + 15)^2}$$

$$= \frac{1500}{(2t + 15)^2}$$

(a) $a(5) = \dfrac{1500}{[2(5) + 15]^2} = 2.4$ ft/sec^2

(b) $a(10) = \dfrac{1500}{[2(10) + 15]^2} \approx 1.2$ ft/sec^2

(c) $a(20) = \dfrac{1500}{[2(20) + 15]^2} \approx 0.5$ ft/sec^2

105. $f(x) = g(x)h(x)$

(a) $f'(x) = g(x)h'(x) + h(x)g'(x)$

$$f''(x) = g(x)h''(x) + g'(x)h'(x) + h(x)g''(x) + h'(x)g'(x)$$

$$= g(x)h''(x) + 2g'(x)h'(x) + h(x)g''(x)$$

$$f'''(x) = g(x)h'''(x) + g'(x)h''(x) + 2g'(x)h''(x) + 2g''(x)h'(x) + h(x)g'''(x) + h'(x)g''(x)$$

$$= g(x)h'''(x) + 3g'(x)h''(x) + 3g''(x)h'(x) + g'''(x)h(x)$$

$$f^{(4)}(x) = g(x)h^{(4)}(x) + g'(x)h'''(x) + 3g'(x)h'''(x) + 3g''(x)h''(x) + 3g''(x)h''(x) + 3g'''(x)h'(x) + g'''(x)h'(x) + g^{(4)}(x)h(x)$$

$$= g(x)h^{(4)}(x) + 4g'(x)h'''(x) + 6g''(x)h''(x) + 4g'''(x)h'(x) + g^{(4)}(x)h(x)$$

(b)
$$f^{(n)}(x) = g(x)h^{(n)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{1[(n-1)(n-2)\cdots(2)(1)]}g'(x)h^{(n-1)}(x) + \frac{n(n-1)(n-2)\cdots(2)(1)}{(2)(1)[(n-2)(n-3)\cdots(2)(1)]}g''(x)h^{(n-2)}(x)$$

$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{(3)(2)(1)[(n-3)(n-4)\cdots(2)(1)]}g'''(x)h^{(n-3)}(x) + \cdots$$

$$+ \frac{n(n-1)(n-2)\cdots(2)(1)}{[(n-1)(n-2)\cdots(2)(1)](1)}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

$$= g(x)h^{(n)}(x) + \frac{n!}{1!(n-1)!}g'(x)h^{(n-1)}(x) + \frac{n!}{2!(n-2)!}g''(x)h^{(n-2)}(x) + \cdots$$

$$+ \frac{n!}{(n-1)!1!}g^{(n-1)}(x)h'(x) + g^{(n)}(x)h(x)$$

Note: $n! = n(n-1)\ldots 3 \cdot 2 \cdot 1$ (read "n factorial.")

107. $f(x) = \cos x \qquad f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

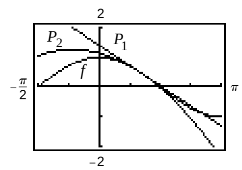
$f''(x) = -\cos x \qquad f''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

(a) $P_1(x) = f'(a)(x - a) + f(a) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a)$

$= -\frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2}$

(b)



(c) P_2 is a better approximation.

(d) The accuracy worsens as you move farther away from $x = a = (\pi/3)$.

109. False. If $y = f(x)g(x)$, then

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x).$$

111. True

$$h'(c) = f(c)g'(c) + g(c)f'(c)$$
$$= f(c)(0) + g(c)(0)$$
$$= 0$$

113. True

115. $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \quad 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

$f'(x) = \left\{\begin{matrix} 2x, & \text{if } x \quad 0 \\ -2x, & \text{if } x < 0 \end{matrix}\right\} = 2|x|$

$f''(x) = \begin{cases} 2, & \text{if } x > 0 \\ -2, & \text{if } x < 0 \end{cases}$

$f''(0)$ does not exist since the left and right derivatives are not equal.

Section 2.4 The Chain Rule

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1.	$y = (6x - 5)^4$	$u = 6x - 5$	$y = u^4$
3.	$y = \sqrt{x^2 - 1}$	$u = x^2 - 1$	$y = \sqrt{u}$
5.	$y = \csc^3 x$	$u = \csc x$	$y = u^3$

7. $y = (2x - 7)^3$

$y' = 3(2x - 7)^2(2) = 6(2x - 7)^2$

9. $g(x) = 3(4 - 9x)^4$

$g'(x) = 12(4 - 9x)^3(-9) = -108(4 - 9x)^3$

11. $f(x) = (9 - x^2)^{2/3}$

$f'(x) = \frac{2}{3}(9 - x^2)^{-1/3}(-2x) = -\frac{4x}{3(9 - x^2)^{1/3}}$

13. $f(t) = (1 - t)^{1/2}$

$f'(t) = \frac{1}{2}(1 - t)^{-1/2}(-1) = -\frac{1}{2\sqrt{1 - t}}$

15. $y = (9x^2 + 4)^{1/3}$

$$y' = \frac{1}{3}(9x^2 + 4)^{-2/3}(18x) = \frac{6x}{(9x^2 + 4)^{2/3}}$$

17. $y = 2(4 - x^2)^{1/4}$

$$y' = 2\left(\frac{1}{4}\right)(4 - x^2)^{-3/4}(-2x)$$

$$= \frac{-x}{\sqrt[4]{(4 - x^2)^3}}$$

19. $y = (x - 2)^{-1}$

$$y' = -1(2 - x)^{-2}(1) = \frac{-1}{(x - 2)^2}$$

21. $f(t) = (t - 3)^{-2}$

$$f'(t) = -2(t - 3)^{-3} = \frac{-2}{(t - 3)^3}$$

23. $y = (x + 2)^{-1/2}$

$$\frac{dy}{dx} = -\frac{1}{2}(x + 2)^{-3/2} = -\frac{1}{2(x + 2)^{3/2}}$$

25. $f(x) = x^2(x - 2)^4$

$$f'(x) = x^2[4(x - 2)^3(1)] + (x - 2)^4(2x)$$

$$= 2x(x - 2)^3[2x + (x - 2)]$$

$$= 2x(x - 2)^3(3x - 2)$$

27. $y = x\sqrt{1 - x^2} = x(1 - x^2)^{1/2}$

$$y' = x\left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] + (1 - x^2)^{1/2}(1)$$

$$= -x^2(1 - x^2)^{-1/2} + (1 - x^2)^{1/2}$$

$$= (1 - x^2)^{-1/2}[-x^2 + (1 - x^2)]$$

$$= \frac{1 - 2x^2}{\sqrt{1 - x^2}}$$

29. $y = \dfrac{x}{\sqrt{x^2 + 1}} = x(x^2 + 1)^{-1/2}$

$$y' = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}(1)$$

$$= -x^2(x^2 + 1)^{-3/2} + (x^2 + 1)^{-1/2}$$

$$= (x^2 + 1)^{-3/2}[-x^2 + (x^2 + 1)]$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

31. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

$$g'(x) = 2\left(\frac{x + 5}{x^2 + 2}\right)\left(\frac{(x^2 + 2) - (x + 5)(2x)}{(x^2 + 2)^2}\right)$$

$$= \frac{2(x + 5)(2 - 10x - x^2)}{(x^2 + 2)^3}$$

33. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

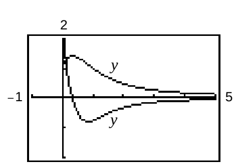
$$f'(v) = 3\left(\frac{1 - 2v}{1 + v}\right)^2\left(\frac{(1 + v)(-2) - (1 - 2v)}{(1 + v)^2}\right)$$

$$= \frac{-9(1 - 2v)^2}{(1 + v)^4}$$

35. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

$$y' = \frac{1 - 3x^2 - 4x^{3/2}}{2\sqrt{x}(x^2 + 1)^2}$$

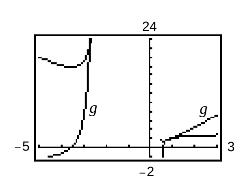
The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



37. $g(t) = \dfrac{3t^2}{\sqrt{t^2 + 2t - 1}}$

$$g'(t) = \frac{3t(t^2 + 3t - 2)}{(t^2 + 2t - 1)^{3/2}}$$

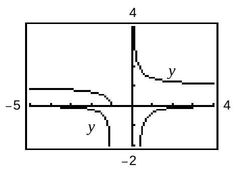
The zeros of g' correspond to the points on the graph of g where the tangent lines are horizontal.



39. $y = \sqrt{\dfrac{x+1}{x}}$

$y' = -\dfrac{\sqrt{(x+1)/x}}{2x(x+1)}$

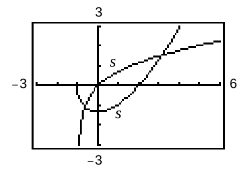
y' has no zeros.



41. $s(t) = \dfrac{-2(2-t)\sqrt{1+t}}{3}$

$s'(t) = \dfrac{t}{\sqrt{1+t}}$

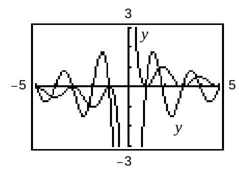
The zero of $s'(t)$ corresponds to the point on the graph of $s(t)$ where the tangent line is horizontal.



43. $y = \dfrac{\cos \pi x + 1}{x}$

$$\frac{dy}{dx} = \frac{-\pi x \sin \pi x - \cos \pi x - 1}{x^2}$$

$$= -\frac{\pi x \sin \pi x + \cos \pi x + 1}{x^2}$$



The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

45. (a) $y = \sin x$

$y' = \cos x$

$y'(0) = 1$

1 cycle in $[0, 2\pi]$

(b) $y = \sin 2x$

$y' = 2 \cos 2x$

$y'(0) = 2$

2 cycles in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a.

47. $y = \cos 3x$

$\dfrac{dy}{dx} = -3 \sin 3x$

49. $g(x) = 3 \tan 4x$

$g'(x) = 12 \sec^2 4x$

51. $y = \sin (\pi x)^2 = \sin (\pi^2 x^2)$

$y' = \cos (\pi x^2)[2\pi^2 x] = 2\pi^2 x \cos(\pi^2 x^2)$

53. $h(x) = \sin 2x \cos 2x$

$h'(x) = \sin 2x(-2 \sin 2x) + \cos 2x(2 \cos 2x)$

$= 2 \cos^2 2x - 2 \sin^2 2x$

$= 2 \cos 4x.$

Alternate solution: $h(x) = \dfrac{1}{2} \sin 4x$

$h'(x) = \dfrac{1}{2} \cos 4x(4) = 2 \cos 4x$

55. $f(x) = \dfrac{\cot x}{\sin x} = \dfrac{\cos x}{\sin^2 x}$

$$f'(x) = \frac{\sin^2 x(-\sin x) - \cos x(2 \sin x \cos x)}{\sin^4 x}$$

$$= \frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x} = \frac{-1 - \cos^2 x}{\sin^3 x}$$

57. $y = 4\sec^2 x$

$y' = 8\sec x \cdot \sec x \tan x = 8\sec^2 x \tan x$

59. $f(\theta) = \frac{1}{4}\sin^2 2\theta = \frac{1}{4}(\sin 2\theta)^2$

$f'(\theta) = 2\left(\frac{1}{4}\right)(\sin 2\theta)(\cos 2\theta)(2)$

$= \sin 2\theta \cos 2\theta = \frac{1}{2}\sin 4\theta$

61. $f(x) = 3\sec^2(\pi t - 1)$

$f'(t) = 6\sec(\pi t - 1)\sec(\pi t - 1)\tan(\pi t - 1)(\pi)$

$= 6\pi \sec^2(\pi t - 1)\tan(\pi t - 1) = \dfrac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)}$

63. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

$= \sqrt{x} + \frac{1}{4}\sin(4x^2)$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} + \dfrac{1}{4}\cos(4x^2)(8x)$

$= \dfrac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

65. $y = \sin(\cos x)$

$\dfrac{dy}{dx} = \cos(\cos x) \cdot (-\sin x)$

$= -\sin x \cos(\cos x)$

67. $s(t) = (t^2 + 2t + 8)^{1/2}, \quad (2, 4)$

$s'(t) = \dfrac{1}{2}(t^2 + 2t + 8)^{-1/2}(2t + 2)$

$= \dfrac{t + 1}{\sqrt{t^2 + 2t + 8}}$

$s'(2) = \dfrac{3}{4}$

69. $f(x) = \dfrac{3}{x^3 - 4} = 3(x^3 - 4)^{-1}, \quad \left(-1, -\dfrac{3}{5}\right)$

$f'(x) = -3(x^3 - 4)^{-2}(3x^2) = -\dfrac{9x^2}{(x^3 - 4)^2}$

$f'(-1) = -\dfrac{9}{25}$

71. $f(t) = \dfrac{3t + 2}{t - 1}, \quad (0, -2)$

$f'(t) = \dfrac{(t - 1)(3) - (3t + 2)(1)}{(t - 1)^2} = \dfrac{-5}{(t - 1)^2}$

$f'(0) = -5$

73. $y = 37 - \sec^3(2x), \quad (0, 36)$

$y' = -3\sec^2(2x)[2\sec(2x)\tan(2x)]$

$= -6\sec^3(2x)\tan(2x)$

$y'(0) = 0$

75. (a) $f(x) = \sqrt{3x^2 - 2}, \quad (3, 5)$

$f'(x) = \dfrac{1}{2}(3x^2 - 2)^{-1/2}(6x)$

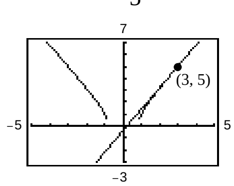
$= \dfrac{3x}{\sqrt{3x^2 - 2}}$

$f'(3) = \dfrac{9}{5}$

Tangent line:

$y - 5 = \dfrac{9}{5}(x - 3) \qquad 9x - 5y - 2 = 0$

(b)



77. (a) $f(x) = \sin 2x, \quad (\pi, 0)$

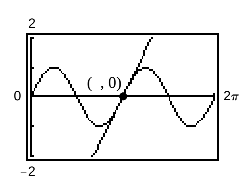
$f'(x) = 2\cos 2x$

$f'(\pi) = 2$

Tangent line:

$y = 2(x - \pi) \qquad 2x - y - 2\pi = 0$

(b)



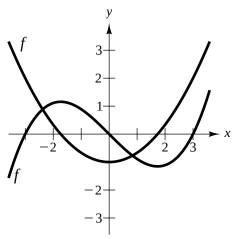
79. $f(x) = 2(x^2 - 1)^3$

$$f'(x) = 6(x^2 - 1)^2(2x)$$
$$= 12x(x^4 - 2x^2 + 1)$$
$$= 12x^5 - 24x^3 + 12x$$
$$f''(x) = 60x^4 - 72x^2 + 12$$
$$= 12(5x^2 - 1)(x^2 - 1)$$

81. $f(x) = \sin x^2$

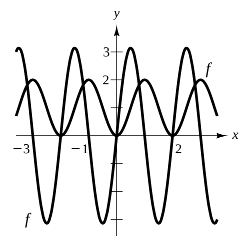
$$f'(x) = 2x \cos x^2$$
$$f''(x) = 2x[2x(-\sin x^2)] + 2\cos x^2$$
$$= 2[\cos x^2 - 2x^2 \sin x^2]$$

83.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

85.



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

87. $g(x) = f(3x)$

$g'(x) = f'(3x)(3) \qquad g'(x) = 3f'(3x)$

89. (a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)h'(x) + g'(x)h(x)$$
$$f'(5) = (-3)(-2) + (6)(3) = 24$$

(b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))h'(x)$$
$$f'(5) = g'(3)(-2) = -2g'(3)$$

Need $g'(3)$ to find $f'(5)$.

(c) $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$$
$$f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \frac{4}{3}$$

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2 g'(x)$$
$$f'(5) = 3(-3)^2(6) = 162$$

91. (a) $f = 132{,}400(331 - v)^{-1}$

$$f' = (-1)(132{,}400)(331 - v)^{-2}(-1)$$
$$= \frac{132{,}400}{(331 - v)^2}$$

When $v = 30$, $f' \approx 1.461$.

(b) $f = 132{,}400(331 + v)^{-1}$

$$f' = (-1)(132{,}400)(331 + v)^{-2}(1)$$
$$= \frac{-132{,}400}{(331 + v)^2}$$

When $v = 30$, $f' \approx -1.016$.

93. $\theta = 0.2 \cos 8t$

The maximum angular displacement is $\theta = 0.2$ (since $-1 \quad \cos 8t \quad 1$).

$$\frac{d\theta}{dt} = 0.2[-8 \sin 8t] = -1.6 \sin 8t$$

When $t = 3$, $d\theta/dt = -1.6 \sin 24 \approx 1.4489$ radians per second.

95. $S = C(R^2 - r^2)$

$$\frac{dS}{dt} = C\left(2R\frac{dR}{dt} - 2r\frac{dr}{dt}\right)$$

Since r is constant, we have $dr/dt = 0$ and

$$\frac{dS}{dt} = (1.76 \times 10^5)(2)(1.2 \times 10^{-2})(10^{-5})$$
$$= 4.224 \times 10^{-2} = 0.04224.$$

97. (a) $x = -1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161$

(b) $C = 60x + 1350$

$$= 60(-1.6372t^3 + 19.3120t^2 - 0.5082t - 0.6161) + 1350$$

$$\frac{dC}{dt} = 60(-4.9116t^2 + 38.624t - 0.5082)$$

$$= -294.696t^2 + 2317.44t - 30.492$$

The function $\frac{dC}{dt}$ is quadratic, not linear. The cost function levels off at the end of the day, perhaps due to fatigue.

99. $f(x) = \sin \beta x$

(a) $f'(x) = \beta \cos \beta x$

$f''(x) = -\beta^2 \sin \beta x$

$f'''(x) = -\beta^3 \cos \beta x$

$f^{(4)} = \beta^4 \sin \beta x$

(b) $f''(x) + \beta^2 f(x) = -\beta^2 \sin \beta x + \beta^2(\sin \beta x) = 0$

(c) $f^{(2k)}(x) = (-1)^k \beta^{2k} \sin \beta x$

$f^{(2k-1)}(x) = (-1)^{k+1} \beta^{2k-1} \cos \beta x$

101. (a) $r'(x) = f'(g(x))g'(x)$

$r'(1) = f'(g(1))g'(1)$

Note that $g(1) = 4$ and $f'(4) = \frac{5-0}{6-2} = \frac{5}{4}$

Also, $g'(1) = 0$. Thus, $r'(1) = 0$

(b) $s'(x) = g'(f(x))f'(x)$

$s'(4) = g'(f(4))f'(4)$

Note that $f(4) = \frac{5}{2}$, $g'\left(\frac{5}{2}\right) = \frac{6-4}{6-2} = \frac{1}{2}$ and

$f'(4) = \frac{5}{4}$.

Thus, $s'(4) = \frac{1}{2}\left(\frac{5}{4}\right) = \frac{5}{8}$.

103. $g = \sqrt{x(x+n)}$

$$= \sqrt{x^2 + nx}$$

$$\frac{dg}{dx} = \frac{1}{2}(x^2 + nx)^{-1/2}(2x + n)$$

$$= \frac{2x + n}{2\sqrt{x^2 + nx}}$$

$$= \frac{(2x + n)/2}{\sqrt{x(x + n)}}$$

$$= \frac{[x + (x + n)]/2}{\sqrt{x(x + n)}}$$

$$= \frac{a}{g}$$

105. $g(x) = |2x - 3|$

$$g'(x) = 2\left(\frac{2x - 3}{|2x - 3|}\right), \quad x \neq \frac{3}{2}$$

107. $h(x) = |x| \cos x$

$$h'(x) = -|x| \sin x + \frac{x}{|x|} \cos x, \quad x \neq 0$$

109. (a) $f(x) = \tan \frac{\pi x}{4}$ $\qquad f(1) = 1$

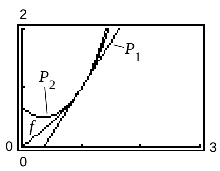
$f'(x) = \frac{\pi}{4}\sec^2 \frac{\pi x}{4}$ $\qquad f'(1) = \frac{\pi}{4}(2) = \frac{\pi}{2}$

$f''(x) = \frac{\pi}{2}\sec^2 \frac{\pi x}{4} \cdot \tan \frac{\pi x}{4}\left(\frac{\pi}{4}\right)$ $\qquad f''(1) = \frac{\pi}{8}(2)(1) = \frac{\pi}{4}$

$P_1(x) = f'(1)(x - 1) + f(1) = \frac{\pi}{2}(x - 1) + 1.$

$P_2(x) = \frac{1}{2}\left(\frac{\pi}{4}\right)(x - 1)^2 + f'(1)(x - 1) + f(1) = \frac{\pi}{8}(x - 1)^2 + \frac{\pi}{2}(x - 1) + 1$

(b)



(c) P_2 is a better approximation than P_1

(d) The accuracy worsens as you move away from $x = c = 1$.

111. False. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}(-1)$.

113. True

Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 36$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$

3. $x^{1/2} + y^{1/2} = 9$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$

5. $x^3 - xy + y^2 = 4$

$3x^2 - xy' - y + 2yy' = 0$

$(2y - x)y' = y - 3x^2$

$y' = \frac{y - 3x^2}{2y - x}$

7. $x^3y^3 - y - x = 0$

$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$

$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$

$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$

9. $x^3 - 3x^2 + 2xy^2 = 12$

$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$

$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$

$y^1 = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$

11. $\sin x + 2\cos 2y = 1$

$\cos x - 4(\sin 2y)y' = 0$

$y' = \frac{\cos x}{4 \sin 2y}$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$

$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

15. $y = \sin(xy)$

$$y' = [xy' + y]\cos(xy)$$

$$y' - x\cos(xy)y' = y\cos(xy)$$

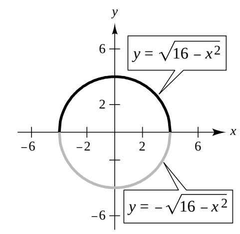
$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

17. (a) $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}(16 - x^2)^{-1/2}(-2x)$$

$$= \frac{\mp x}{\sqrt{16 - x^2}} = \frac{-x}{\pm\sqrt{16 - x^2}} = \frac{-x}{y}$$

(d) Implicitly:

$$2x + 2yy' = 0$$

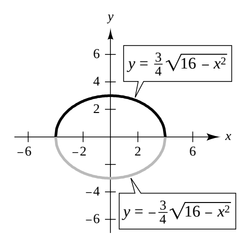
$$y' = -\frac{x}{y}$$

19. (a) $16y^2 = 144 - 9x^2$

$$y^2 = \frac{1}{16}(144 - 9x^2) = \frac{9}{16}(16 - x^2)$$

$$y = \pm\frac{3}{4}\sqrt{16 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{3}{8}(16 - x^2)^{-1/2}(-2x)$$

$$= \mp\frac{3x}{4\sqrt{16 - x^2}} = \frac{-3x}{4(4/3)y} = \frac{-9x}{16y}$$

(d) Implicitly:

$$18x + 32yy' = 0$$

$$y' = \frac{-9x}{16y}$$

21. $xy = 4$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

At $(-4, -1)$: $y' = -\frac{1}{4}$

23. $y^2 = \dfrac{x^2 - 4}{x^2 + 4}$

$$2yy' = \frac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2}$$

$$2yy' = \frac{16x}{(x^2 + 4)^2}$$

$$y' = \frac{8x}{y(x^2 + 4)^2}$$

At $(2, 0)$, y' is undefined.

25. $$x^{2/3} + y^{2/3} = 5$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}}$$

At $(8, 1)$: $y' = -\frac{1}{2}$.

27. $$\tan(x + y) = x$$

$$(1 + y')\sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} = -\sin^2(x + y)$$

$$= -\frac{x^2}{x^2 + 1}$$

At $(0, 0)$: $y' = 0$.

29. $$(x^2 + 4)y = 8$$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At $(2, 1)$: $y' = \frac{-32}{64} = -\frac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \frac{8}{x^2 + 4}\right)$

31. $$(x^2 + y^2)^2 = 4x^2y$$

$$2(x^2 + y^2)(2x + 2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At $(1, 1)$: $y' = 0$.

33. $$\tan y = x$$

$$y'\sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, \ -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

35. $$x^2 + y^2 = 36$$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} = \frac{-36}{y^3}$$

37. $$x^2 - y^2 = 16$$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$x - yy' = 0$$

$$1 - yy'' - (y')^2 = 0$$

$$1 - yy'' - \left(\frac{x}{y}\right)^2 = 0$$

$$y^2 - y^3y'' = x^2$$

$$y'' = \frac{y^2 - x^2}{y^3} = \frac{-16}{y^3}$$

39. $$y^2 = x^3$$

$$2yy' = 3x^2$$

$$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$$

$$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$$

$$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2}$$

$$= \frac{3y}{4x^2} = \frac{3x}{4y}$$

41. $\sqrt{x} + \sqrt{y} = 4$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

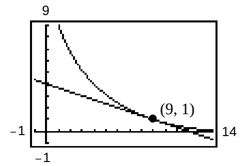
$y' = \frac{-\sqrt{y}}{\sqrt{x}}$

At (9, 1), $y' = -\frac{1}{3}$

Tangent line: $y - 1 = -\frac{1}{3}(x - 9)$

$y = -\frac{1}{3}x + 4$

$x + 3y - 12 = 0$



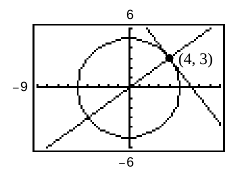
43. $x^2 + y^2 = 25$

$y' = \frac{-x}{y}$

At (4, 3):

Tangent line: $y - 3 = \frac{-4}{3}(x - 4)$ $\quad 4x + 3y - 25 = 0$

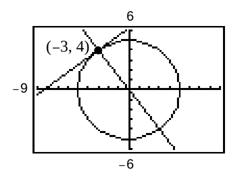
Normal line: $y - 3 = \frac{3}{4}(x - 4)$ $\quad 3x - 4y = 0.$



At (−3, 4):

Tangent line: $y - 4 = \frac{3}{4}(x + 3)$ $\quad 3x - 4y + 25 = 0$

Normal line: $y - 4 = \frac{-4}{3}(x + 3)$ $\quad 4x + 3y = 0.$



45. $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$ = slope of tangent line

$\frac{y}{x}$ = slope of normal line

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

47. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$$50x + 32yy' + 200 - 160y' = 0$$

$$y' = \frac{200 + 50x}{160 - 32y}$$

Horizontal tangents occur when $x = -4$:

$$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$$

$$y(y - 10) = 0 \qquad y = 0, 10$$

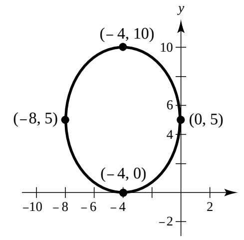
Horizontal tangents: $(-4, 0), (-4, 10)$.

Vertical tangents occur when $y = 5$:

$$25x^2 + 400 + 200x - 800 + 400 = 0$$

$$25x(x + 8) = 0 \qquad x = 0, -8$$

Vertical tangents: $(0, 5), (-8, 5)$.



49. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.

Ellipse:	*Parabola:*
$4x + 2yy' = 0$	$2yy' = 4$
$y' = -\frac{2x}{y}$	$y' = \frac{2}{y}$

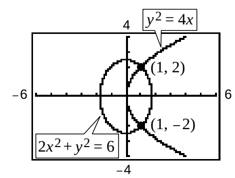
At $(1, 2)$, the slopes are:

$y' = -1$ $\qquad$ $y' = 1$.

At $(1, -2)$, the slopes are:

$y' = 1$ $\qquad$ $y' = -1$.

Tangents are perpendicular.



51. $y = -x$ and $x = \sin y$

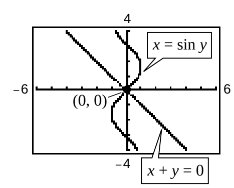
Point of intersection: $(0, 0)$

$y = -x$:	$x = \sin y$:
$y' = -1$	$1 = y' \cos y$
	$y' = \sec y$

At $(0, 0)$, the slopes are:

$y' = -1$ $\qquad$ $y' = 1$.

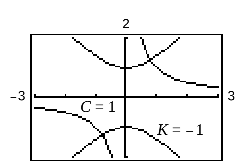
Tangents are perpendicular.

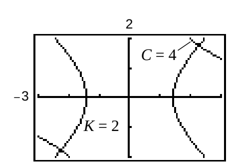


53.

$xy = C$	$x^2 - y^2 = K$
$xy' + y = 0$	$2x - 2yy' = 0$
$y' = -\frac{y}{x}$	$y' = \frac{x}{y}$

At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.





55. $2y^2 - 3x^4 = 0$

(a) $4yy' - 12x^3 = 0$

$4yy' = 12x^3$

$y' = \dfrac{12x^3}{4y} = \dfrac{3x^3}{y}$

(b) $4y\dfrac{dy}{dt} - 12x^3\dfrac{dx}{dt} = 0$

$y\dfrac{dy}{dt} = 3x^3\dfrac{dx}{dt}$

57. $\cos \pi y - 3\sin \pi x = 1$

(a) $-\pi \sin(\pi y)y' - 3\pi\cos \pi x = 0$

$y' = \dfrac{-3\cos \pi x}{\sin \pi y}$

(b) $-\pi \sin(\pi y)\dfrac{dy}{dt} - 3\pi \cos(\pi x)\dfrac{dx}{dt} = 0$

$-\sin(\pi y)\dfrac{dy}{dt} = 3\cos(\pi x)\dfrac{dx}{dt}$

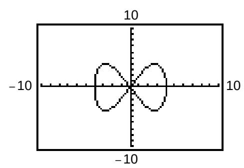
59. A function is in explicit form if y is written as a function of x: $y = f(x)$. For example, $y = x^3$. An implicit equation is not in the form $y = f(x)$. For example, $x^2 + y^2 = 5$.

61. (a) $x^4 = 4(4x^2 - y^2)$

$4y^2 = 16x^2 - x^4$

$y^2 = 4x^2 - \dfrac{1}{4}x^4$

$y = \pm\sqrt{4x^2 - \dfrac{1}{4}x^4}$



(b) $y = 3 \qquad 9 = 4x^2 - \dfrac{1}{4}x^4$

$36 = 16x^2 - x^4$

$x^4 - 16x^2 + 36 = 0$

$x^2 = \dfrac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$

Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = \left(1 \pm \sqrt{7}\right)^2$.

Hence, there are four values of x:

$-1-\sqrt{7},\ 1-\sqrt{7},\ -1+\sqrt{7},\ 1+\sqrt{7}$

To find the slope, $2yy' = 8x - x^3 \qquad y' = \dfrac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$y_1 = \frac{1}{3}\left(\sqrt{7} + 7\right)\left(x + 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + 8\sqrt{7} + 23\right]$.

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

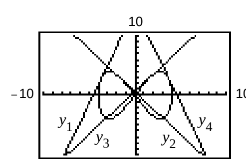
$y_2 = \frac{1}{3}\left(\sqrt{7} - 7\right)\left(x - 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} - 7\right)x + 23 - 8\sqrt{7}\right]$.

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$y_3 = -\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right]$.

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$y_4 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right]$.



—CONTINUED—

61. —CONTINUED—

(c) Equating y_3 and y_4,

$$-\frac{1}{3}\left(\sqrt{7}-7\right)\left(x+1-\sqrt{7}\right)+3=-\frac{1}{3}\left(\sqrt{7}+7\right)\left(x-1-\sqrt{7}\right)+3$$

$$\left(\sqrt{7}-7\right)\left(x+1-\sqrt{7}\right)=\left(\sqrt{7}+7\right)\left(x-1-\sqrt{7}\right)$$

$$\sqrt{7}x+\sqrt{7}-7-7x-7+7\sqrt{7}=\sqrt{7}x-\sqrt{7}-7+7x-7-7\sqrt{7}$$

$$16\sqrt{7}=14x$$

$$x=\frac{8\sqrt{7}}{7}$$

If $x=\frac{8\sqrt{7}}{7}$, then $y=5$ and the lines intersect at $\left(\frac{8\sqrt{7}}{7}, 5\right)$.

63. Let $f(x)=x^n=x^{p/q}$, where p and q are nonzero integers and $q>0$. First consider the case where $p=1$. The derivative of $f(x)=x^{1/q}$ is given by

$$\frac{d}{dx}\left[x^{1/q}\right]=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

where $t=x+\Delta x$. Observe that

$$\frac{f(t)-f(x)}{t-x}=\frac{t^{1/q}-x^{1/q}}{t-x}=\frac{t^{1/q}-x^{1/q}}{(t^{1/q})^q-(x^{1/q})^q}$$

$$=\frac{t^{1/q}-x^{1/q}}{(t^{1/q}-x^{1/q})(t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdot\cdot\cdot+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)})}$$

$$=\frac{1}{t^{1-(1/q)}+t^{1-(2/q)}x^{1/q}+\cdot\cdot\cdot+t^{1/q}x^{1-(2/q)}+x^{1-(1/q)}}.$$

As $t\to x$, the denominator approaches $qx^{1-(1/q)}$. That is,

$$\frac{d}{dx}\left[x^{1/q}\right]=\frac{1}{qx^{1-(1/q)}}=\frac{1}{q}x^{(1/q)-1}.$$

Now consider $f(x)=x^{p/q}=(x^p)^{1/q}$. From the Chain Rule,

$$f'(x)=\frac{1}{q}(x^p)^{(1/q)-1}\frac{d}{dx}[x^p]=\frac{1}{q}(x^p)^{(1/q)-1}px^{p-1}=\frac{p}{q}x^{[(p/q)-p]+(p-1)}=\frac{p}{q}x^{(p/q)-1}=nx^{n-1}\left(n=\frac{p}{q}\right).$$

Section 2.6 Related Rates

1. $y=\sqrt{x}$

$$\frac{dy}{dt}=\left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt}=2\sqrt{x}\,\frac{dy}{dt}$$

(a) When $x=4$ and $dx/dt=3$,

$$\frac{dy}{dt}=\frac{1}{2\sqrt{4}}(3)=\frac{3}{4}.$$

(b) When $x=25$ and $dy/dt=2$,

$$\frac{dx}{dt}=2\sqrt{25}\,(2)=20.$$

3. $xy=4$

$$x\frac{dy}{dt}+y\frac{dx}{dt}=0$$

$$\frac{dy}{dt}=\left(-\frac{y}{x}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt}=\left(-\frac{x}{y}\right)\frac{dy}{dt}$$

(a) When $x=8$, $y=1/2$, and $dx/dt=10$,

$$\frac{dy}{dt}=-\frac{1/2}{8}(10)=-\frac{5}{8}.$$

(b) When $x=1$, $y=4$, and $dy/dt=-6$,

$$\frac{dx}{dt}=-\frac{1}{4}(-6)=\frac{3}{2}.$$

5. $y = x^2 + 1$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = 2x\frac{dx}{dt}$

(a) When $x = -1$,

$$\frac{dy}{dt} = 2(-1)(2) = -4 \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 2(0)(2) = 0 \text{ cm/sec.}$$

(c) When $x = 1$,

$$\frac{dy}{dt} = 2(1)(2) = 4 \text{ cm/sec.}$$

7. $y = \tan x$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = \sec^2 x\frac{dx}{dt}$

(a) When $x = -\pi/3$,

$$\frac{dy}{dt} = (2)^2(2) = 8 \text{ cm/sec.}$$

(b) When $x = -\pi/4$,

$$\frac{dy}{dt} = \left(\sqrt{2}\right)^2(2) = 4 \text{ cm/sec.}$$

(c) When $x = 0$,

$$\frac{dy}{dt} = (1)^2(2) = 2 \text{ cm/sec.}$$

9. (a) $\frac{dx}{dt}$ negative $\quad \frac{dy}{dt}$ positive

(b) $\frac{dy}{dt}$ positive $\quad \frac{dx}{dt}$ negative

11. Yes, y changes at a constant rate: $\frac{dy}{dt} = a \cdot \frac{dx}{dt}$.

No, the rate $\frac{dy}{dt}$ is a multiple of $\frac{dx}{dt}$.

13. $D = \sqrt{x^2 + y^2} = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^4 + 3x^2 + 1}$

$\frac{dx}{dt} = 2$

$$\frac{dD}{dt} = \frac{1}{2}(x^4 + 3x^2 + 1)^{-1/2}(4x^3 + 6x)\frac{dx}{dt} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 + 1}}\frac{dx}{dt} = \frac{4x^3 + 6x}{\sqrt{x^4 + 3x^2 + 1}}$$

15. $A = \pi r^2$

$\frac{dr}{dt} = 3$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

(a) When $r = 6$,

$$\frac{dA}{dt} = 2\pi(6)(3) = 36\pi \text{ cm}^2\text{/min.}$$

(b) When $r = 24$,

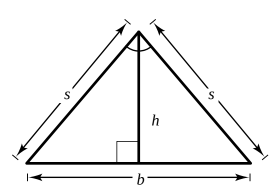
$$\frac{dA}{dt} = 2\pi(24)(3) = 144\pi \text{ cm}^2\text{/min.}$$

17. (a) $\sin\frac{\theta}{2} = \frac{(1/2)b}{s} \qquad b = 2s\sin\frac{\theta}{2}$

$\cos\frac{\theta}{2} = \frac{h}{s} \qquad h = s\cos\frac{\theta}{2}$

$$A = \frac{1}{2}bh = \frac{1}{2}\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)$$

$$= \frac{s^2}{2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{s^2}{2}\sin\theta$$



(b) $\frac{dA}{dt} = \frac{s^2}{2}\cos\theta\frac{d\theta}{dt}$ where $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min.

When $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}s^2}{8}$

When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = \frac{s^2}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{s^2}{8}$

(c) If $d\theta/dt$ is constant, dA/dt is proportional to $\cos\theta$.

19. $V = \frac{4}{3}\pi r^3, \ \frac{dV}{dt} = 800$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\left(\frac{dV}{dt}\right) = \frac{1}{4\pi r^2}(800)$$

(a) When $r = 30$, $\frac{dr}{dt} = \frac{1}{4\pi(30)^2}(800) = \frac{2}{9\pi}$ cm/min.

(b) When $r = 60$, $\frac{dr}{dt} = \frac{1}{4\pi(60)^2}(800) = \frac{1}{18\pi}$ cm/min.

21. $s = 6x^2$

$$\frac{dx}{dt} = 3$$

$$\frac{ds}{dt} = 12x\frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{ds}{dt} = 12(1)(3) = 36 \text{ cm}^2/\text{sec}.$$

(b) When $x = 10$,

$$\frac{ds}{dt} = 12(10)(3) = 360 \text{ cm}^2/\text{sec}.$$

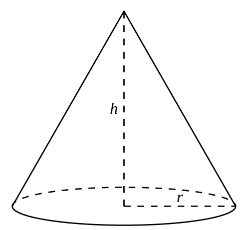
23. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{9}{4}h^2\right)h \quad [\text{since } 2r = 3h]$

$$= \frac{3\pi}{4}h^3$$

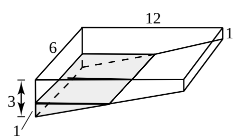
$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{9\pi}{4}h^2\frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{4(dV/dt)}{9\pi h^2}$$

When $h = 15$, $\frac{dh}{dt} = \frac{4(10)}{9\pi(15)^2} = \frac{8}{405\pi}$ ft/min.



25.

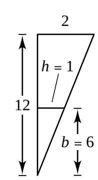


(a) Total volume of pool $= \frac{1}{2}(2)(12)(6) + (1)(6)(12) = 144 \text{ m}^3$

Volume of 1m. of water $= \frac{1}{2}(1)(6)(6) = 18 \text{ m}^3$

(see similar triangle diagram)

% pool filled $= \frac{18}{144}(100\%) = 12.5\%$



(b) Since for 0 h 2, $b = 6h$, you have

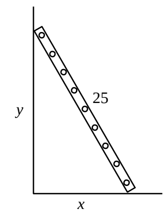
$$V = \frac{1}{2}bh(6) = 3bh = 3(6h)h = 18h^2$$

$$\frac{dV}{dt} = 36h\frac{dh}{dt} = \frac{1}{4} \qquad \frac{dh}{dt} = \frac{1}{144h} = \frac{1}{144(1)} = \frac{1}{144} \text{ m/min.}$$

27. $x^2 + y^2 = 25^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y}\cdot\frac{dx}{dt} = \frac{-2x}{y} \text{ since } \frac{dx}{dt} = 2.$$



(a) When $x = 7, y = \sqrt{576} = 24, \frac{dy}{dt} = \frac{-2(7)}{24} = \frac{-7}{12}$ ft/sec.

When $x = 15, y = \sqrt{400} = 20, \frac{dy}{dt} = \frac{-2(15)}{20} = \frac{-3}{2}$ ft/sec.

When $x = 24, y = 7, \frac{dy}{dt} = \frac{-2(24)}{7} = \frac{-48}{7}$ ft/sec.

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

From part (a) we have $x = 7, y = 24, \frac{dx}{dt} = 2$, and $\frac{dy}{dt} = -\frac{7}{12}$.

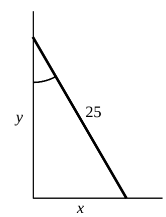
Thus, $\frac{dA}{dt} = \frac{1}{2}\left[7\left(-\frac{7}{12}\right) + 24(2)\right]$

$$= \frac{527}{24} \approx 21.96 \text{ ft}^2\text{/sec.}$$

(c) $\tan\theta = \frac{x}{y}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}$$

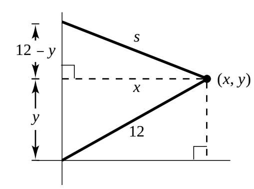
$$\frac{d\theta}{dt} = \cos^2\theta\left[\frac{1}{y}\cdot\frac{dx}{dt} - \frac{x}{y^2}\cdot\frac{dy}{dt}\right]$$



Using $x = 7, y = 24, \frac{dx}{dt} = 2, \frac{dy}{dt} = -\frac{7}{12}$ and $\cos\theta = \frac{24}{25}$, we have $\frac{d\theta}{dt} = \left(\frac{24}{25}\right)^2\left[\frac{1}{24}(2) - \frac{7}{(24)^2}\left(-\frac{7}{12}\right)\right] = \frac{1}{12}$ rad/sec.

29. When $y = 6, x = \sqrt{12^2 - 6^2} = 6\sqrt{3}$, and

$$s = \sqrt{x^2 + (12 - y)^2}$$
$$= \sqrt{108 + 36} = 12.$$



$$x^2 + (12 - y)^2 = s^2$$

$$2x\frac{dx}{dt} + 2(12 - y)(-1)\frac{dy}{dt} = 2s\frac{ds}{dt}$$

$$x\frac{dx}{dt} + (y - 12)\frac{dy}{dt} = s\frac{ds}{dt}$$

Also, $x^2 + y^2 = 12^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus, $x\frac{dx}{dt} + (y - 12)\left(\frac{-x}{y}\frac{dx}{dt}\right) = s\frac{ds}{dt}$

$$\frac{dx}{dt}\left[x - x + \frac{12x}{y}\right] = s\frac{ds}{dt} \qquad \frac{dx}{dt} = \frac{sy}{12x}\cdot\frac{ds}{dt} = \frac{(12)(6)}{(12)(6\sqrt{3})}(-0.2) = \frac{-1}{5\sqrt{3}} = \frac{-\sqrt{3}}{15} \text{ m/sec (horizontal)}$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt} = \frac{-6\sqrt{3}}{6}\cdot\frac{(-\sqrt{3})}{15} = \frac{1}{5} \text{ m/sec (vertical).}$$

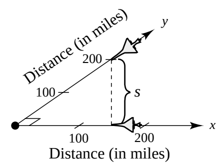
31. (a) $s^2 = x^2 + y^2$

$$\frac{dx}{dt} = -450$$

$$\frac{dy}{dt} = -600$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{x(dx/dt) + y(dy/dt)}{s}$$



When $x = 150$ and $y = 200$, $s = 250$ and

$$\frac{ds}{dt} = \frac{150(-450) + 200(-600)}{250} = -750 \text{ mph.}$$

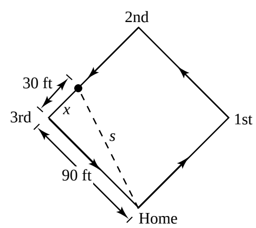
(b) $t = \frac{250}{750} = \frac{1}{3}\text{ hr} = 20 \text{ min}$

33. $s^2 = 90^2 + x^2$

$$x = 30$$

$$\frac{dx}{dt} = -28$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \qquad \frac{ds}{dt} = \frac{x}{s}\cdot\frac{dx}{dt}$$



When $x = 30$,

$$s = \sqrt{90^2 + 30^2} = 30\sqrt{10}$$

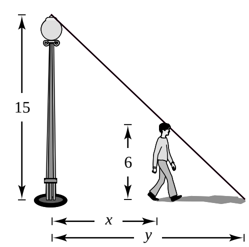
$$\frac{ds}{dt} = \frac{30}{30\sqrt{10}}(-28) = \frac{-28}{\sqrt{10}} \approx -8.85 \text{ ft/sec.}$$

35. (a) $\frac{15}{6} = \frac{y}{y - x} \qquad 15y - 15x = 6y$

$$y = \frac{5}{3}x$$

$$\frac{dx}{dt} = 5$$

$$\frac{dy}{dt} = \frac{5}{3}\cdot\frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec}$$



(b) $\frac{d(y - x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{25}{3} - 5 = \frac{10}{3} \text{ ft/sec}$

37. $x(t) = \frac{1}{2}\sin\frac{\pi t}{6}, x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi/6} = 12$ seconds

(b) When $x = \frac{1}{2}, y = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ m.

Lowest point: $\left(0, \frac{\sqrt{3}}{2}\right)$

(c) When $x = \frac{1}{4}, y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and $t = 1$

$$\frac{dx}{dt} = \frac{1}{2}\left(\frac{\pi}{6}\right)\cos\frac{\pi t}{6} = \frac{\pi}{12}\cos\frac{\pi t}{6}$$

$$x^2 + y^2 = 1$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

Thus,

$$\frac{dy}{dt} = -\frac{1/4}{\sqrt{15}/4}\cdot\frac{\pi}{12}\cos\left(\frac{\pi}{6}\right)$$
$$= \frac{-\pi}{\sqrt{15}}\left(\frac{1}{12}\right)\frac{\sqrt{3}}{2} = \frac{-\pi}{24}\frac{1}{\sqrt{5}} = \frac{-\sqrt{5}\pi}{120}.$$

$$\text{Speed} = \left|\frac{-\sqrt{5}\pi}{120}\right| = \frac{\sqrt{5}\pi}{120}\text{ m/sec}$$

39. Since the evaporation rate is proportional to the surface area, $dV/dt = k(4\pi r^2)$. However, since $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Therefore,

$$k(4\pi r^2) = 4\pi r^2\frac{dr}{dt} \qquad k = \frac{dr}{dt}.$$

41.

$$pV^{1.3} = k$$

$$1.3\,pV^{0.3}\frac{dV}{dt} + V^{1.3}\frac{dp}{dt} = 0$$

$$V^{0.3}\left(1.3p\frac{dV}{dt} + V\frac{dp}{dt}\right) = 0$$

$$1.3p\frac{dV}{dt} = -V\frac{dp}{dt}$$

43.

$$\tan\theta = \frac{y}{30}$$

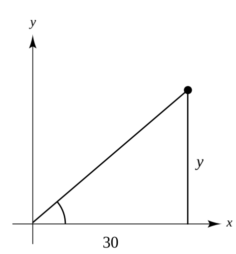
$$\frac{dy}{dt} = 3\text{ m/sec.}$$

$$\sec^2\theta\cdot\frac{d\theta}{dt} = \frac{1}{30}\frac{dy}{dt}$$

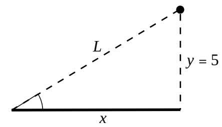
$$\frac{d\theta}{dt} = \frac{1}{30}\cos^2\theta\cdot\frac{dy}{dt}$$

When $y = 30$, $\theta = \pi/4$ and $\cos\theta = \sqrt{2}/2$. Thus,

$$\frac{d\theta}{dt} = \frac{1}{30}\left(\frac{1}{2}\right)(3) = \frac{1}{20}\text{ rad/sec.}$$



45. $\tan \theta = \dfrac{y}{x}, y = 5$



$$\frac{dx}{dt} = -600 \text{ mi/hr}$$

$$(\sec^2 \theta)\frac{d\theta}{dt} = -\frac{5}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2 \theta\left(-\frac{5}{x^2}\right)\frac{dx}{dt} = \frac{x^2}{L^2}\left(-\frac{5}{x^2}\right)\frac{dx}{dt}$$

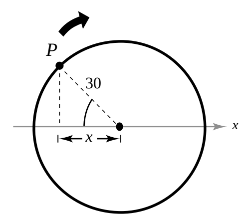
$$= \left(-\frac{5^2}{L^2}\right)\left(\frac{1}{5}\right)\frac{dx}{dt} = (-\sin^2 \theta)\left(\frac{1}{5}\right)(-600) = 120 \sin^2 \theta$$

(a) When $\theta = 30°, \dfrac{d\theta}{dt} = \dfrac{120}{4} = 30$ rad/hr $= \dfrac{1}{2}$ rad/min.

(b) When $\theta = 60°, \dfrac{d\theta}{dt} = 120\left(\dfrac{3}{4}\right) = 90$ rad/hr $= \dfrac{3}{2}$ rad/min.

(c) When $\theta = 75°, \dfrac{d\theta}{dt} = 120 \sin^2 75° \approx 111.96$ rad/hr ≈ 1.87 rad/min.

47. $\dfrac{d\theta}{dt} = (10 \text{ rev/sec})(2\pi \text{ rad/rev}) = 20\pi \text{ rad/sec}$

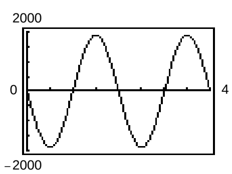


(a) $\cos \theta = \dfrac{x}{30}$

$$-\sin \theta \frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = -30 \sin \theta \frac{d\theta}{dt} = -30 \sin \theta(20\pi) = -600\pi \sin \theta$$

(b)



(c) $|dx/dt| = |-600\pi \sin \theta|$ is greatest when $\sin \theta = 1 \qquad \theta = (\pi/2) + n\pi$ (or $90° + n \cdot 180°$)

$|dx/dt|$ is least when $\theta = n\pi$ (or $n \cdot 180°$).

(d) For $\theta = 30°, \dfrac{dx}{dt} = -600\pi \sin(30°) = -600\pi\dfrac{1}{2} = -300\pi$ cm/sec.

For $\theta = 60°, \dfrac{dx}{dt} = -600\pi \sin(60°) = -600\pi\dfrac{\sqrt{3}}{2} = -300\sqrt{3}\,\pi$ cm/sec

49. $\tan \theta = \dfrac{x}{50} \qquad x = 50 \tan \theta$

$$\frac{dx}{dt} = 50 \sec^2 \theta \ \frac{d\theta}{dt}$$

$$2 = 50 \sec^2 \theta \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\cos^2 \theta, -\frac{\pi}{4} \quad \theta \quad \frac{\pi}{4}$$

51. $x^2 + y^2 = 25$; acceleration of the top of the ladder $= \dfrac{d^2y}{dt^2}$

First derivative: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Second derivative: $x\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt}\cdot\dfrac{dx}{dt} + y\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt}\cdot\dfrac{dy}{dt} = 0$

$$\frac{d^2y}{dt^2} = \left(\frac{1}{y}\right)\left[-x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right]$$

When $x = 7, y = 24, \dfrac{dy}{dt} = -\dfrac{7}{12}$, and $\dfrac{dx}{dt} = 2$ (see Exercise 27). Since $\dfrac{dx}{dt}$ is constant, $\dfrac{d^2x}{dt^2} = 0$.

$$\frac{d^2y}{dt^2} = \frac{1}{24}\left[-7(0) - (2)^2 - \left(-\frac{7}{12}\right)^2\right] = \frac{1}{24}\left[-4 - \frac{49}{144}\right] = \frac{1}{24}\left[-\frac{625}{144}\right] \approx -0.1808 \text{ ft/sec}^2$$

53. (a) Using a graphing utility, you obtain $m(s) = -0.881s^2 + 29.10s - 206.2$

(b) $\dfrac{dm}{dt} = \dfrac{dm}{ds}\,\dfrac{ds}{dt} = (-1.762s + 29.10)\dfrac{ds}{dt}$

(c) If $t = s$ (1995), then $s = 15.5$ and $\dfrac{ds}{dt} = 1.2$.

Thus, $\dfrac{dm}{dt} = (-1.762(15.5) + 29.10)(1.2) \approx 2.15$ million.

Review Exercises for Chapter 2

1. $f(x) = x^2 - 2x + 3$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 - 2(x + \Delta x) + 3] - [x^2 - 2x + 3]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + 3) - (x^2 - 2x + 3)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 - 2(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2
\end{aligned}$$

3. $f(x) = \sqrt{x} + 1$

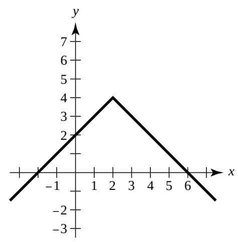
$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt{x + \Delta x} + 1) - (\sqrt{x} + 1)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

5. f is differentiable for all $x \neq -1$.

7. $f(x) = 4 - |x - 2|$

(a) Continuous at $x = 2$.

(b) Not differentiable at $x = 2$ because of the sharp turn in the graph.



9. Using the limit definition, you obtain $g'(x) = \frac{4}{3}x - \frac{1}{6}$.

At $x = -1$, $g'(-1) = -\frac{4}{3} - \frac{1}{6} = \frac{-3}{2}$

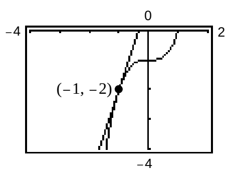
11. (a) Using the limit defintion, $f'(x) = 3x^2$.

At $x = -1$, $f'(-1) = 3$. The tangent line is

$y - (-2) = 3(x - (-1))$

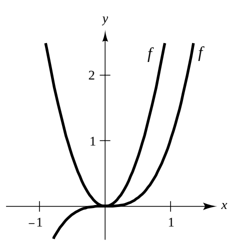
$y = 3x + 1$

(b)



13.
$$\begin{aligned} g'(2) &= \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} \\ &= \lim_{x \to 2} \frac{x^2(x - 1) - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{x^3 - x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x^2 + x + 2)}{x - 2} \\ &= \lim_{x \to 2} (x^2 + x + 2) = 8 \end{aligned}$$

15.



17. $y = 25$

$y' = 0$

19. $f(x) = x^8$

$f'(x) = 8x^7$

21. $h(t) = 3t^4$

$h'(t) = 12t^3$

23. $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

25. $h(x) = 6\sqrt{x} + 3\sqrt[3]{x} = 6x^{1/2} + 3x^{1/3}$

$h'(x) = 3x^{-1/2} + x^{-2/3} = \frac{3}{\sqrt{x}} + \frac{1}{\sqrt[3]{x^2}}$

27. $g(t) = \frac{2}{3}t^{-2}$

$g'(x) = \frac{-4}{3}t^{-3} = \frac{-4}{3t^3}$

29. $f(\theta) = 2\theta - 3\sin\theta$

$f'(\theta) = 2 - 3\cos\theta$

31. $f(\theta) = 3\cos\theta - \frac{\sin\theta}{4}$

$f'(\theta) = -3\sin\theta - \frac{\cos\theta}{4}$

33. $F = 200\sqrt{T}$

$F'(t) = \dfrac{100}{\sqrt{T}}$

(a) When $T = 4$, $F'(4) = 50$ vibrations/sec/lb.

(b) When $T = 9$, $F'(9) = 33\frac{1}{3}$ vibrations/sec/lb.

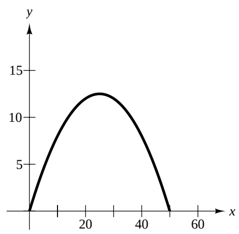
35. $s(t) = -16t^2 + s_0$

$s(9.2) = -16(9.2)^2 + s_0 = 0$

$s_0 = 1354.24$

The building is approximately 1354 feet high (or 415 m).

37. (a)



Total horizontal distance: 50

(b) $0 = x - 0.02x^2$

$0 = x\left(1 - \dfrac{x}{50}\right)$ implies $x = 50$.

(c) Ball reaches maximum height when $x = 25$.

(d) $y = x - 0.02x^2$

$y' = 1 - 0.04x$

$y'(0) = 1$

$y'(10) = 0.6$

$y'(25) = 0$

$y'(30) = -0.2$

$y'(50) = -1$

(e) $y'(25) = 0$

39. $x(t) = t^2 - 3t + 2 = (t - 2)(t - 1)$

(a) $v(t) = x'(t) = 2t - 3$

$a(t) = v'(t) = 2$

(b) $v(t) < 0$ for $t < \frac{3}{2}$.

(c) $v(t) = 0$ for $t = \frac{3}{2}$.

$x = \left(\frac{3}{2} - 2\right)\left(\frac{3}{2} - 1\right) = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4}$

(d) $x(t) = 0$ for $t = 1, 2$.

$|v(1)| = |2(1) - 3| = 1$

$|v(2)| = |2(2) - 3| = 1$

The speed is 1 when the position is 0.

41. $f(x) = (3x^2 + 7)(x^2 - 2x + 3)$

$f'(x) = (3x^2 + 7)(2x - 2) + (x^2 - 2x + 3)(6x)$

$= 2(6x^3 - 9x^2 + 16x - 7)$

43. $h(x) = \sqrt{x}\sin x = x^{1/2}\sin x$

$h'(x) = \dfrac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$

45. $f(x) = 2x - x^{-2}$

$f'(x) = 2 + 2x^{-3} = 2\left(1 + \dfrac{1}{x^3}\right)$

$= \dfrac{2(x^3 + 1)}{x^3}$

47. $f(x) = \dfrac{x^2 + x - 1}{x^2 - 1}$

$f'(x) = \dfrac{(x^2 - 1)(2x + 1) - (x^2 + x - 1)(2x)}{(x^2 - 1)^2}$

$= \dfrac{-(x^2 + 1)}{(x^2 - 1)^2}$

49. $f(x) = (4 - 3x^2)^{-1}$

$f'(x) = -(4 - 3x^2)^{-2}(-6x) = \dfrac{6x}{(4 - 3x^2)^2}$

51. $y = \dfrac{x^2}{\cos x}$

$y' = \dfrac{\cos x\,(2x) - x^2(-\sin x)}{\cos^2 x} = \dfrac{2x\cos x + x^2\sin x}{\cos^2 x}$

53. $y = 3x^2\sec x$

$y' = 3x^2\sec x\tan x + 6x\sec x$

55. $y = -x\tan x$

$y' = -x\sec^2 x - \tan x$

57. $y = x\cos x - \sin x$

$y' = -x\sin x + \cos x - \cos x = -x\sin x$

59. $g(t) = t^3 - 3t + 2$

$g'(t) = 3t^2 - 3$

$g''(t) = 6t$

61. $f(\theta) = 3\tan\theta$

$f'(\theta) = 3\sec^2\theta$

$f''(\theta) = 6\sec\theta(\sec\theta\tan\theta) = 6\sec^2\theta\tan\theta$

63. $y = 2\sin x + 3\cos x$

$y' = 2\cos x - 3\sin x$

$y'' = -2\sin x - 3\cos x$

$y'' + y = -(2\sin x + 3\cos x) + (2\sin x + 3\cos x)$

$= 0$

65. $f(x) = (1 - x^3)^{1/2}$

$f'(x) = \frac{1}{2}(1 - x^3)^{-1/2}(-3x^2)$

$= -\frac{3x^2}{2\sqrt{1 - x^3}}$

67. $h(x) = \left(\frac{x - 3}{x^2 + 1}\right)^2$

$h'(x) = 2\left(\frac{x - 3}{x^2 + 1}\right)\left(\frac{(x^2 + 1)(1) - (x - 3)(2x)}{(x^2 + 1)^2}\right)$

$= \frac{2(x - 3)(-x^2 + 6x + 1)}{(x^2 + 1)^3}$

69. $f(s) = (s^2 - 1)^{5/2}(s^3 + 5)$

$f'(s) = (s^2 - 1)^{5/2}(3s^2) + (s^3 + 5)\left(\frac{5}{2}\right)(s^2 - 1)^{3/2}(2s)$

$= s(s^2 - 1)^{3/2}[3s(s^2 - 1) + 5(s^3 + 5)]$

$= s(s^2 - 1)^{3/2}(8s^3 - 3s + 25)$

71. $y = 3\cos(3x + 1)$

$y' = -9\sin(3x + 1)$

73. $y = \frac{1}{2}\csc 2x$

$y' = \frac{1}{2}(-\csc 2x\cot 2x)(2)$

$= -\csc 2x\cot 2x$

75. $y = \frac{x}{2} - \frac{\sin 2x}{4}$

$y' = \frac{1}{2} - \frac{1}{4}\cos 2x(2)$

$= \frac{1}{2}(1 - \cos 2x) = \sin^2 x$

77. $y = \frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x$

$y' = \sin^{1/2} x\cos x - \sin^{5/2} x\cos x$

$= (\cos x)\sqrt{\sin x}(1 - \sin^2 x)$

$= (\cos^3 x)\sqrt{\sin x}$

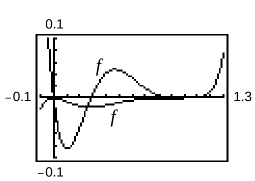
79. $y = \frac{\sin \pi x}{x + 2}$

$y' = \frac{(x + 2)\pi\cos\pi x - \sin\pi x}{(x + 2)^2}$

81. $f(t) = t^2(t - 1)^5$

$f'(t) = t(t - 1)^4(7t - 2)$

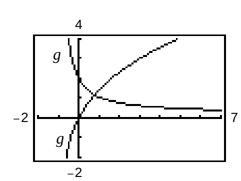
The zeros of f' correspond to the points on the graph of f where the tangent line is horizontal.



83. $g(x) = 2x(x + 1)^{-1/2}$

$g'(x) = \frac{x + 2}{(x + 1)^{3/2}}$

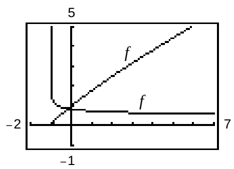
g' does not equal zero for any value of x in the domain. The graph of g has no horizontal tangent lines.



85. $f(t) = (t + 1)^{1/2}(t + 1)^{1/3} = (t + 1)^{5/6}$

$$f'(t) = \frac{5}{6(t + 1)^{1/6}}$$

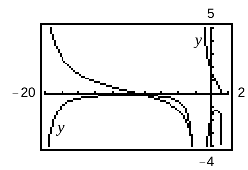
f' does not equal zero for any x in the domain. The graph of f has no horizontal tangent lines.



87. $y = \tan\sqrt{1 - x}$

$$y' = -\frac{\sec^2\sqrt{1 - x}}{2\sqrt{1 - x}}$$

y' does not equal zero for any x in the domain. The graph has no horizontal tangent lines.



89. $y = 2x^2 + \sin 2x$

$y' = 4x + 2\cos 2x$

$y'' = 4 - 4\sin 2x$

91. $f(x) = \cot x$

$f'(x) = -\csc^2 x$

$f'' = -2\csc x(-\csc x \cdot \cot x)$

$= 2\csc^2 x \cot x$

93. $f(t) = \dfrac{t}{(1 - t)^2}$

$$f'(t) = \frac{t + 1}{(1 - t)^3}$$

$$f''(t) = \frac{2(t + 2)}{(1 - t)^4}$$

95. $g(\theta) = \tan 3\theta - \sin(\theta - 1)$

$g'(\theta) = 3\sec^2 3\theta - \cos(\theta - 1)$

$g''(\theta) = 18\sec^2 3\theta \tan 3\theta + \sin(\theta - 1)$

97. $T = 700(t^2 + 4t + 10)^{-1}$

$$T' = \frac{-1400(t + 2)}{(t^2 + 4t + 10)^2}$$

(a) When $t = 1$,

$$T' = \frac{-1400(1 + 2)}{(1 + 4 + 10)^2} \approx -18.667 \text{ deg/hr.}$$

(b) When $t = 3$,

$$T' = \frac{-1400(3 + 2)}{(9 + 12 + 10)^2} \approx -7.284 \text{ deg/hr.}$$

(c) When $t = 5$,

$$T' = \frac{-1400(5 + 2)}{(25 + 30 + 10)^2} \approx -3.240 \text{ deg/hr.}$$

(d) When $t = 10$,

$$T' = \frac{-1400(10 + 2)}{(100 + 40 + 10)^2} \approx -0.747 \text{ deg/hr.}$$

99. $x^2 + 3xy + y^3 = 10$

$2x + 3xy' + 3y + 3y^2y' = 0$

$3(x + y^2)y' = -(2x + 3y)$

$$y' = \frac{-(2x + 3y)}{3(x + y^2)}$$

101. $y\sqrt{x} - x\sqrt{y} = 16$

$$y\left(\frac{1}{2}x^{-1/2}\right) + x^{1/2}y' - x\left(\frac{1}{2}y^{-1/2}y'\right) - y^{1/2} = 0$$

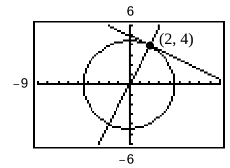
$$\left(\sqrt{x} - \frac{x}{2\sqrt{y}}\right)y' = \sqrt{y} - \frac{y}{2\sqrt{x}}$$

$$\frac{2\sqrt{xy} - x}{2\sqrt{y}}y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}}$$

$$y' = \frac{2\sqrt{xy} - y}{2\sqrt{x}} \cdot \frac{2\sqrt{y}}{2\sqrt{xy} - x} = \frac{2y\sqrt{x} - y\sqrt{y}}{2x\sqrt{y} - x\sqrt{x}}$$

103.
$$x \sin y = y \cos x$$
$$(x \cos y)y' + \sin y = -y \sin x + y' \cos x$$
$$y'(x \cos y - \cos x) = -y \sin x - \sin y$$
$$y' = \frac{y \sin x + \sin y}{\cos x - x \cos y}$$

105. $x^2 + y^2 = 20$



$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

At $(2, 4)$: $y' = -\frac{1}{2}$

Tangent line: $y - 4 = -\frac{1}{2}(x - 2)$
$$x + 2y - 10 = 0$$

Normal line: $y - 4 = 2(x - 2)$
$$2x - y = 0$$

107. $y = \sqrt{x}$

$\frac{dy}{dt} = 2$ units/sec

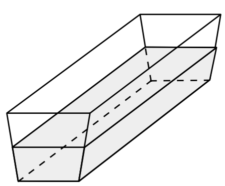
$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt} \qquad \frac{dx}{dt} = 2\sqrt{x}\frac{dy}{dt} = 4\sqrt{x}$

(a) When $x = \frac{1}{2}$, $\frac{dx}{dt} = 2\sqrt{2}$ units/sec.

(b) When $x = 1$, $\frac{dx}{dt} = 4$ units/sec.

(c) When $x = 4$, $\frac{dx}{dt} = 8$ units/sec.

109. $\frac{s}{h} = \frac{1/2}{2}$



$s = \frac{1}{4}h$

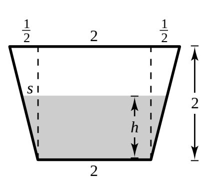
$\frac{dV}{dt} = 1$

Width of water at depth h:

$$w = 2 + 2s = 2 + 2\left(\frac{1}{4}h\right) = \frac{4 + h}{2}$$

$$V = \frac{5}{2}\left(2 + \frac{4 + h}{2}\right)h = \frac{5}{4}(8 + h)h$$

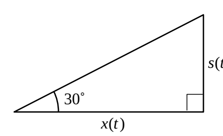
$$\frac{dV}{dt} = \frac{5}{2}(4 + h)\frac{dh}{dt}$$



$$\frac{dh}{dt} = \frac{2(dV/dt)}{5(4 + h)}$$

When $h = 1$, $\frac{dh}{dt} = \frac{2}{25}$ m/min.

111. $s(t) = 60 - 4.9t^2$



$$s'(t) = -9.8t$$
$$s = 35 = 60 - 4.9t^2$$
$$4.9t^2 = 25$$
$$t = \frac{5}{\sqrt{4.9}}$$
$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{s(t)}{x(t)}$$
$$x(t) = \sqrt{3}s(t)$$
$$\frac{dx}{dt} = \sqrt{3}\frac{ds}{dt} = \sqrt{3}(-9.8)\frac{5}{\sqrt{4.9}}$$
$$\approx -38.34 \text{ m/sec}$$