

CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

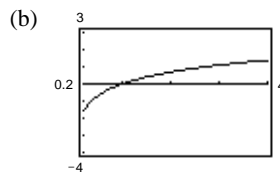
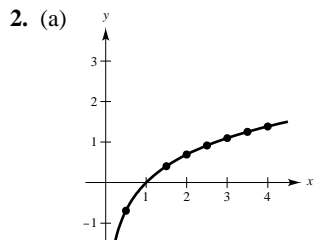
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CHAPTER 5

Logarithmic, Exponential, and Other Transcendental Functions

Section 5.1 The Natural Logarithmic Function: Differentiation

Solutions to Even-Numbered Exercises



The graphs are identical.

4. (a) $\ln 8.3 \approx 2.1163$

(b) $\int_1^{8.3} \frac{1}{t} dt \approx 2.1163$

6. (a) $\ln 0.6 \approx -0.5108$

(b) $\int_1^{0.6} \frac{1}{t} dt \approx -0.5108$

8. $f(x) = -\ln x$

Reflection in the x -axis

Matches (d)

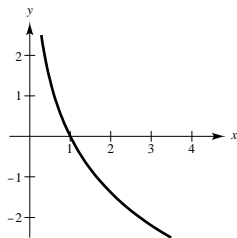
10. $f(x) = -\ln(-x)$

Reflection in the y -axis and the x -axis

Matches (c)

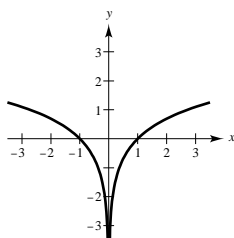
12. $f(x) = -2 \ln x$

Domain: $x > 0$



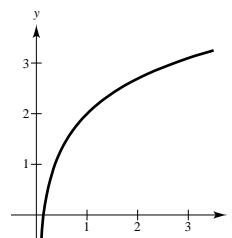
14. $f(x) = \ln|x|$

Domain: $x \neq 0$



16. $g(x) = 2 + \ln x$

Domain: $x > 0$



18. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - 2 \ln 2 \approx -1.3862$

(b) $\ln 24 = 3 \ln 2 + \ln 3 \approx 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}(2 \ln 2 + \ln 3) \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - (3 \ln 2 + 2 \ln 3) \approx -4.2765$

20. $\ln \sqrt{2^3} = \ln 2^{3/2} = \frac{3}{2} \ln 2$

22. $\ln xyz = \ln x + \ln y + \ln z$

24. $\ln \sqrt{a-1} = \ln(a-1)^{1/2} = \left(\frac{1}{2}\right) \ln(a-1)$

26. $\ln 3e^2 = \ln 3 + 2 \ln e = 2 + \ln 3$

28. $\ln \frac{1}{e} = \ln 1 - \ln e = -1$

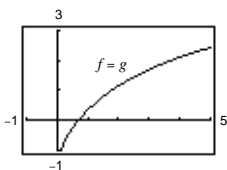
$$30. 3 \ln x + 2 \ln y - 4 \ln z = \ln x^3 + \ln y^2 - \ln z^4$$

$$= \ln \frac{x^3 y^2}{z^4}$$

$$32. 2[\ln x - \ln(x+1) - \ln(x-1)] = 2 \ln \frac{x}{(x+1)(x-1)} = \ln \left(\frac{x}{x^2-1} \right)^2$$

$$34. \frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)] = \frac{3}{2} \ln \frac{x^2+1}{(x+1)(x-1)} = \ln \sqrt{\left(\frac{x^2+1}{x^2-1} \right)^3}$$

36.



$$38. \lim_{x \rightarrow 6^-} \ln(6-x) = -\infty$$

$$40. \lim_{x \rightarrow 5^+} \ln \frac{x}{\sqrt{x-4}} = \ln 5 \approx 1.6094$$

$$42. y = \ln x^{3/2} = \frac{3}{2} \ln x$$

$$y' = \frac{3}{2x}$$

$$\text{At } (1, 0), y' = \frac{3}{2}.$$

$$44. y = \ln x^{1/2} = \frac{1}{2} \ln x$$

$$y' = \frac{1}{2x}$$

$$\text{At } (1, 0), y' = \frac{1}{2}.$$

$$46. h(x) = \ln(2x^2 + 1)$$

$$h'(x) = \frac{1}{2x^2+1}(4x) = \frac{4x}{2x^2+1}$$

$$48. y = x \ln x$$

$$\frac{dy}{dx} = x \left(\frac{1}{x} \right) + \ln x = 1 + \ln x$$

$$50. y = \ln \sqrt{x^2-4} = \frac{1}{2} \ln(x^2-4)$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{2x}{x^2-4} \right) = \frac{x}{x^2-4}$$

$$52. f(x) = \ln \left(\frac{2x}{x+3} \right) = \ln 2x - \ln(x+3)$$

$$f'(x) = \frac{1}{x} - \frac{1}{x+3} = \frac{3}{x(x+3)}$$

$$54. h(t) = \frac{\ln t}{t}$$

$$h'(t) = \frac{t(1/t) - \ln t}{t^2} = \frac{1 - \ln t}{t^2}$$

$$56. y = \ln(\ln x)$$

$$\frac{dy}{dx} = \frac{1/x}{\ln x} = \frac{1}{x \ln x}$$

$$58. y = \ln \sqrt[3]{\frac{x-1}{x+1}} = \frac{1}{3} [\ln(x-1) - \ln(x+1)]$$

$$y' = \frac{1}{3} \left[\frac{1}{x-1} - \frac{1}{x+1} \right] = \frac{1}{3} \frac{2}{x^2-1} = \frac{2}{3(x^2-1)}$$

$$60. f(x) = \ln(x + \sqrt{4+x^2})$$

$$f'(x) = \frac{1}{x + \sqrt{4+x^2}} \left(1 + \frac{x}{\sqrt{4+x^2}} \right)$$

$$= \frac{1}{\sqrt{4+x^2}}$$

$$62. \quad y = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln\left(\frac{2+\sqrt{x^2+4}}{x}\right) = \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4} \ln(2+\sqrt{x^2+4}) + \frac{1}{4} \ln x$$

$$\frac{dy}{dx} = \frac{-2x^2(x/\sqrt{x^2+4}) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4} \left(\frac{1}{2+\sqrt{x^2+4}} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x}$$

Note that:

$$\frac{1}{2+\sqrt{x^2+4}} = \frac{1}{2+\sqrt{x^2+4}} \cdot \frac{2-\sqrt{x^2+4}}{2-\sqrt{x^2+4}} = \frac{2-\sqrt{x^2+4}}{-x^2}$$

$$\begin{aligned} \text{Hence, } \frac{dy}{dx} &= \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4} \left(\frac{2-\sqrt{x^2+4}}{-x^2} \right) \left(\frac{x}{\sqrt{x^2+4}} \right) + \frac{1}{4x} \\ &= \frac{-1 + (1/2)(2-\sqrt{x^2+4})}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} \\ &= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} = \frac{\sqrt{x^2+4}}{x^3} \end{aligned}$$

$$64. \quad y = \ln|\csc x|$$

$$y' = \frac{-\csc x \cdot \cot x}{\csc x} = -\cot x$$

$$66. \quad y = \ln|\sec x + \tan x|$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x \end{aligned}$$

$$68. \quad y = \ln\sqrt{1+\sin^2 x} = \frac{1}{2} \ln(1+\sin^2 x)$$

$$\frac{dy}{dx} = \left(\frac{1}{2}\right) \frac{2 \sin x \cos x}{1+\sin^2 x} = \frac{\sin x \cos x}{1+\sin^2 x}$$

$$70. \quad g(x) = \int_1^{\ln x} (t^2 + 3) dt$$

$$g'(x) = [(\ln x)^2 + 3] \frac{d}{dx}(\ln x) = \frac{(\ln x)^2 + 3}{x}$$

(Second Fundamental Theorem of Calculus)

$$72. \quad (a) \quad y = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$$

$$\begin{aligned} \frac{dy}{dx} &= -2x - \frac{1}{(1/2)x + 1} \left(\frac{1}{2}\right) \\ &= -2x - \frac{1}{x+2} \end{aligned}$$

$$\text{When } x = 0, \frac{dy}{dx} = -\frac{1}{2}.$$

$$\text{Tangent line: } y - 4 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 4$$

$$74. \quad \ln(xy) + 5x = 30$$

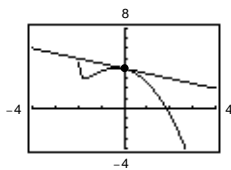
$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y} \frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y+5xy}{x}\right)$$

(b)



76. $y = x(\ln x) - 4x$

$$y' = x\left(\frac{1}{x}\right) + \ln x - 4 = -3 + \ln x$$

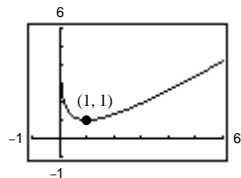
$$(x + y) - xy' = x + x \ln x - 4x - x(-3 + \ln x) = 0$$

78. $y = x - \ln x$

Domain: $x > 0$

$$y' = 1 - \frac{1}{x} = 0 \text{ when } x = 1.$$

$$y'' = \frac{1}{x^2} > 0$$

 Relative minimum: $(1, 1)$


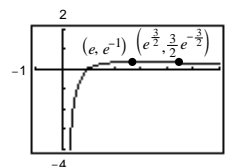
80. $y = \frac{\ln x}{x}$

Domain: $x > 0$

$$y' = \frac{x(1/x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{x^2(-1/x) - (1 - \ln x)(2x)}{x^4} = \frac{2(\ln x) - 3}{x^3} = 0 \text{ when } x = e^{3/2}.$$

 Relative maximum: (e, e^{-1})

 Point of inflection: $(e^{3/2}, \frac{3}{2}e^{-3/2})$


82. $y = x^2 \ln \frac{x}{4}$. Domain $x > 0$

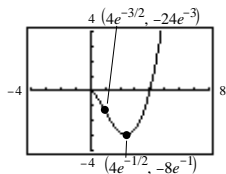
$$y' = x^2\left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2 \ln \frac{x}{4}\right) = 0 \text{ when}$$

$$-1 = 2 \ln \frac{x}{4} \quad \ln \frac{x}{4} = -\frac{1}{2} \quad x = 4e^{-1/2}$$

$$y'' = 1 + 2 \ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2 \ln \frac{x}{4}$$

$$y'' = 0 \text{ when } x = 4e^{-3/2}$$

 Relative minimum: $(4e^{-1/2}, -8e^{-1})$

 Point of inflection: $(4e^{-3/2}, -24e^{-3})$


84. $f(x) = x \ln x$, $f(1) = 0$

$$f'(x) = 1 + \ln x$$
, $f'(1) = 1$

$$f''(x) = \frac{1}{x}$$
, $f''(1) = 1$

$$P_1(x) = f(1) + f'(1)(x - 1) = x - 1$$
, $P_1(1) = 0$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2$$

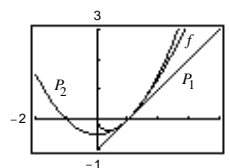
$$= (x - 1) + \frac{1}{2}(x - 1)^2$$
, $P_2(1) = 0$

$$P_1'(x) = 1$$
, $P_1'(1) = 1$

$$P_2'(x) = 1 + (x - 1) = x$$
, $P_2'(1) = 1$

$$P_2''(x) = x$$
, $P_2''(1) = 1$

The values of f , P_1 , P_2 , and their first derivatives agree at $x = 1$. The values of the second derivatives of f and P_2 agree at $x = 1$.



86. Find
- x
- such that
- $\ln x = 3 - x$
- .

$$f(x) = x + (\ln x) - 3 = 0$$

$$f'(x) = 1 + \frac{1}{x}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \left[\frac{4 - \ln x_n}{1 + x_n} \right]$$

n	1	2	3
x_n	2	2.2046	2.2079
$f(x_n)$	-0.3069	-0.0049	0.0000

Approximate root: $x = 2.208$

90.
$$y = \sqrt{\frac{x^2 - 1}{x^2 + 1}}$$

$$\ln y = \frac{1}{2} [\ln(x^2 - 1) - \ln(x^2 + 1)]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[\frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \right]$$

$$\frac{dy}{dx} = \sqrt{\frac{x^2 - 1}{x^2 + 1}} \left[\frac{2x}{x^2 - 1} \right]$$

$$= \frac{(x^2 - 1)^{1/2} 2x}{(x^2 + 1)^{1/2} (x^2 - 1)(x^2 + 1)}$$

$$= \frac{2x}{(x^2 + 1)^{3/2} (x^2 - 1)^{1/2}}$$

94. The base of the natural logarithmic function is
- e
- .

- 96.
- $g(x) = \ln f(x)$
- ,
- $f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

- (a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Since $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and thus, $f'(x) > 0$. Therefore, the graph of f is increasing.

88.
$$y = \sqrt{(x-1)(x-2)(x-3)}$$

$$\ln y = \frac{1}{2} [\ln(x-1) + \ln(x-2) + \ln(x-3)]$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} \right]$$

$$= \frac{1}{2} \left[\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} \right]$$

$$\frac{dy}{dx} = \frac{3x^2 - 12x + 11}{2y}$$

$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}}$$

92.
$$y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

$$\ln y = \ln(x+1) + \ln(x+2) - \ln(x-1) - \ln(x-2)$$

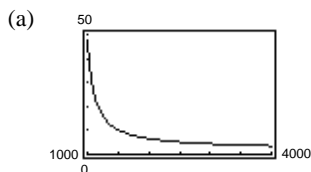
$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x-1} - \frac{1}{x-2}$$

$$\frac{dy}{dx} = y \left[\frac{-2}{x^2-1} + \frac{-4}{x^2-4} \right] = y \left[\frac{-6x^2 + 12}{(x^2-1)(x^2-4)} \right]$$

$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{-6(x^2-2)}{(x+1)(x-1)(x+2)(x-2)}$$

$$= -\frac{6(x^2-2)}{(x-1)^2(x-2)^2}$$

98.
$$t = \frac{5.315}{-6.7968 + \ln x}, 1000 < x$$



(b) $t(1167.41) \approx 20$ years

$$T = (1167.41)(20)(12) = \$280,178.40$$

(c) $t(1068.45) \approx 30$ years

$$T = (1068.45)(30)(12) = \$384,642.00$$

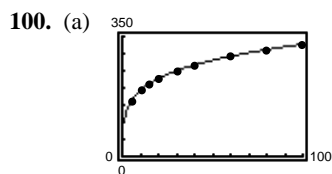
(d)
$$\frac{dt}{dx} = -5.315(-6.7968 + \ln x)^{-2} \left(\frac{1}{x} \right)$$

$$= -\frac{5.315}{x(-6.7968 + \ln x)^2}$$

When $x = 1167.41$, $dt/dx \approx -0.0645$. When $x = 1068.45$, $dt/dx \approx -0.1585$.

- (e) There are two obvious benefits to paying a higher monthly payment:

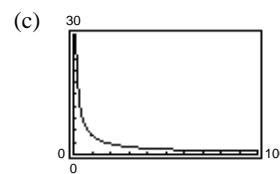
1. The term is lower
2. The total amount paid is lower.



(b) $T'(p) = \frac{34.96}{p} + \frac{3.955}{\sqrt{p}}$

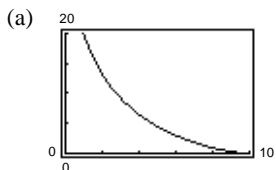
$$T'(10) \approx 4.75 \text{ deg/lb/in}^2$$

$$T'(70) \approx 0.97 \text{ deg/lb/in}^2$$



$$\lim_{p \rightarrow \infty} T'(p) = 0$$

102. $y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2} = 10 [\ln(10 + \sqrt{100 - x^2}) - \ln x] - \sqrt{100 - x^2}$



(c) $\lim_{x \rightarrow 10^-} \frac{dy}{dx} = 0$

(b)
$$\begin{aligned} \frac{dy}{dx} &= 10 \left[\frac{-x}{\sqrt{100 - x^2}(10 + \sqrt{100 - x^2})} - \frac{1}{x} \right] + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} + \frac{x}{\sqrt{100 - x^2}} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{-10}{10 + \sqrt{100 - x^2}} + 1 \right] - \frac{10}{x} \\ &= \frac{x}{\sqrt{100 - x^2}} \left[\frac{\sqrt{100 - x^2}}{10 + \sqrt{100 - x^2}} \right] - \frac{10}{x} \\ &= \frac{x}{10 + \sqrt{100 - x^2}} - \frac{10}{x} \\ &= \frac{x(10 - \sqrt{100 - x^2})}{x^2} - \frac{10}{x} = -\frac{\sqrt{100 - x^2}}{x} \end{aligned}$$

When $x = 5$, $dy/dx = -\sqrt{3}$. When $x = 9$, $dy/dx = -\sqrt{19}/9$.

104. $y = \ln x$

$$y' = \frac{1}{x} > 0 \text{ for } x > 0.$$

Since $\ln x$ is increasing on its entire domain $(0, \infty)$, it is a strictly monotonic function and therefore, is one-to-one.

106. False

π is a constant.

$$\frac{d}{dx} [\ln \pi] = 0$$

Section 5.2 The Natural Logarithmic Function: Integration

2. $\int \frac{10}{x} dx = 10 \int \frac{1}{x} dx = 10 \ln|x| + C$

6.
$$\begin{aligned} \int \frac{1}{3x+2} dx &= \frac{1}{3} \int \frac{1}{3x+2} (3) dx \\ &= \frac{1}{3} \ln|3x+2| + C \end{aligned}$$

4. $u = x - 5, du = dx$

$$\int \frac{1}{x-5} dx = \ln|x-5| + C$$

8. $u = 3 - x^3, du = -3x^2 dx$

$$\begin{aligned} \int \frac{x^2}{3-x^3} dx &= -\frac{1}{3} \int \frac{1}{3-x^3} (-3x^2) dx \\ &= -\frac{1}{3} \ln|3-x^3| + C \end{aligned}$$

10. $u = 9 - x^2, du = -2x dx$

$$\int \frac{x}{\sqrt{9-x^2}} dx = -\frac{1}{2} \int (9-x^2)^{-1/2} (-2x) dx = -\sqrt{9-x^2} + C$$

12. $\int \frac{x(x+2)}{x^3+3x^2-4} dx = \frac{1}{3} \int \frac{3x^2+6x}{x^3+3x^2-4} dx \quad (u = x^3+3x^2-4)$

$$= \frac{1}{3} \ln|x^3+3x^2-4| + C$$

14. $\int \frac{2x^2+7x-3}{x-2} dx = \int \left(2x+11+\frac{19}{x-2}\right) dx$

$$= x^2+11x+19 \ln|x-2| + C$$

16. $\int \frac{x^3-6x-20}{x+5} dx = \int \left(x^2-5x+19-\frac{115}{x+5}\right) dx$

$$= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115 \ln|x+5| + C$$

18. $\int \frac{x^3-3x^2+4x-9}{x^2+3} dx = \int \left(-3+x+\frac{x}{x^2+3}\right) dx$

$$= -3x + \frac{x^2}{2} + \frac{1}{2} \ln(x^2+3) + C$$

20. $\int \frac{1}{x \ln(x^3)} dx = \frac{1}{3} \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$$= \frac{1}{3} \ln|\ln|x|| + C$$

22. $u = 1 + x^{1/3}, du = \frac{1}{3x^{2/3}} dx$

$$\int \frac{1}{x^{2/3}(1+x^{1/3})} dx = 3 \int \frac{1}{1+x^{1/3}} \left(\frac{1}{3x^{2/3}}\right) dx$$

$$= 3 \ln|1+x^{1/3}| + C$$

24. $\int \frac{x(x-2)}{(x-1)^3} dx = \int \frac{x^2-2x+1-1}{(x-1)^3} dx$

$$= \int \frac{(x-1)^2}{(x-1)^3} dx - \int \frac{1}{(x-1)^3} dx$$

$$= \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^3} dx$$

$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

26. $u = 1 + \sqrt{3x}, du = \frac{3}{2\sqrt{3x}} dx \quad dx = \frac{2}{3}(u-1) du$

$$\int \frac{1}{1+\sqrt{3x}} dx = \int \frac{1}{u} \frac{2}{3}(u-1) du$$

$$= \frac{2}{3} \int \left(1 - \frac{1}{u}\right) du$$

$$= \frac{2}{3} [u - \ln|u|] + C$$

$$= \frac{2}{3} [1 + \sqrt{3x} - \ln(1 + \sqrt{3x})] + C$$

$$= \frac{2}{3} \sqrt{3x} - \frac{2}{3} \ln(1 + \sqrt{3x}) + C_1$$

$$28. u = x^{1/3} - 1, du = \frac{1}{3x^{2/3}} dx \quad dx = 3(u + 1)^2 du$$

$$\begin{aligned} \int \frac{\sqrt[3]{x}}{\sqrt[3]{x} - 1} dx &= \int \frac{u + 1}{u} 3(u + 1)^2 du \\ &= 3 \int \frac{u + 1}{u} (u^2 + 2u + 1) du \\ &= 3 \int \left(u^2 + 3u + 3 + \frac{1}{u} \right) du \\ &= 3 \left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u| \right] + C \\ &= 3 \left[\frac{(x^{1/3} - 1)^3}{3} + \frac{3(x^{1/3} - 1)^2}{2} + 3(x^{1/3} - 1) + \ln|x^{1/3} - 1| \right] + C \\ &= 3 \ln|x^{1/3} - 1| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C_1 \end{aligned}$$

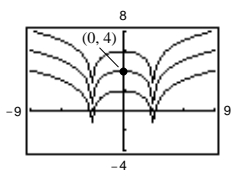
$$30. \int \tan 5\theta d\theta = \frac{1}{5} \int \frac{5 \sin 5\theta}{\cos 5\theta} d\theta \\ = -\frac{1}{5} \ln|\cos 5\theta| + C$$

$$32. \int \sec \frac{x}{2} dx = 2 \int \sec \frac{x}{2} \left(\frac{1}{2} \right) dx \\ = 2 \ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + C$$

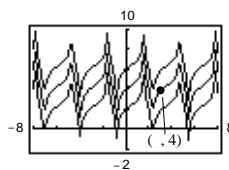
$$34. u = \cot t, du = -\csc^2 t dt \\ \int \frac{\csc^2 t}{\cot t} dt = -\ln|\cot t| + C$$

$$36. \int (\sec t + \tan t) dt = \ln|\sec t + \tan t| - \ln|\cos t| + C \\ = \ln \left| \frac{\sec t + \tan t}{\cos t} \right| + C \\ = \ln|\sec t(\sec t + \tan t)| + C$$

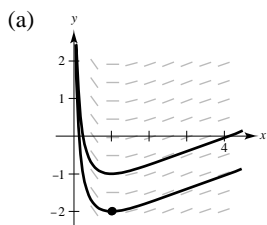
$$38. y = \int \frac{2x}{x^2 - 9} dx \\ = \ln|x^2 - 9| + C \\ (0, 4): 4 = \ln|0 - 9| + C \quad C = 4 - \ln 9 \\ y = \ln|x^2 - 9| + 4 - \ln 9$$



$$40. r = \int \frac{\sec^2 t}{\tan t + 1} dt \\ = \ln|\tan t + 1| + C \\ (\pi, 4): 4 = \ln|0 + 1| + C \quad C = 4 \\ r = \ln|\tan t + 1| + 4$$



$$42. \frac{dy}{dx} = \frac{\ln x}{x}, (1, -2)$$



(b)

$$y = \int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + C \\ y(1) = -2 \quad -2 = \frac{(\ln 1)^2}{2} + C \quad C = -2 \\ \text{Hence, } y = \frac{(\ln x)^2}{2} - 2.$$

$$44. \int_{-1}^1 \frac{1}{x+2} dx = \left[\ln|x+2| \right]_{-1}^1 \\ = \ln 3 - \ln 1 = \ln 3$$

$$46. u = \ln x, du = \frac{1}{x} dx \\ \int_e^{e^2} \frac{1}{x \ln x} dx = \int_e^{e^2} \left(\frac{1}{\ln x} \right) \frac{1}{x} dx = \left[\ln|\ln|x|| \right]_e^{e^2} = \ln 2$$

$$48. \int_0^1 \frac{x-1}{x+1} dx = \int_0^1 1 dx + \int_0^1 \frac{-2}{x+1} dx \\ = \left[x - 2 \ln|x+1| \right]_0^1 = 1 - 2 \ln 2$$

$$50. \int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2 d\theta = \int_{0.1}^{0.2} (\csc^2 2\theta - 2 \csc 2\theta \cot 2\theta + \cot^2 2\theta) d\theta \\ = \int_{0.1}^{0.2} (2 \csc^2 2\theta - 2 \csc 2\theta \cot 2\theta - 1) d\theta \\ = \left[-\cot 2\theta + \csc 2\theta - \theta \right]_{0.1}^{0.2} \approx 0.0024$$

$$52. \ln|\sin x| + C = \ln \left| \frac{1}{\csc x} \right| + C = -\ln|\csc x| + C$$

$$54. -\ln|\csc x + \cot x| + C = -\ln \left| \frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)} \right| + C = -\ln \left| \frac{\csc^2 x - \cot^2 x}{\csc x - \cot x} \right| + C \\ = -\ln \left| \frac{1}{\csc x - \cot x} \right| + C = \ln|\csc x - \cot x| + C$$

$$56. \int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx = -(1 + \sqrt{x})^2 + 6(1 + \sqrt{x}) - 4 \ln(1 + \sqrt{x}) + C_1 \\ = 4\sqrt{x} - x - 4 \ln(1 + \sqrt{x}) + C \text{ where } C = C_1 + 5.$$

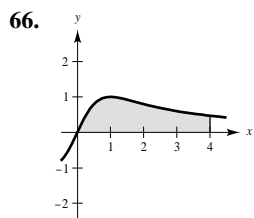
$$58. \int \frac{\tan^2 2x}{\sec 2x} dx = \frac{1}{2} [\ln|\sec 2x + \tan 2x| - \sin 2x] + C$$

$$60. \int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x} dx = \left[\ln|\sec x + \tan x| - 2 \sin x \right]_{-\pi/4}^{\pi/4} \\ = \ln \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right) - 2\sqrt{2} \approx -1.066$$

Note: In Exercises 62 and 64, you can use the Second Fundamental Theorem of Calculus or integrate the function.

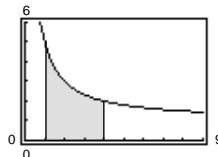
$$62. F(x) = \int_0^x \tan t dt \\ F'(x) = \tan x$$

$$64. F(x) = \int_1^{x^2} \frac{1}{t} dt \\ F'(x) = \frac{2x}{x^2} = \frac{2}{x}$$

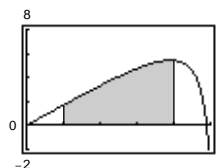


$A \approx 3$
Matches (a)

$$\begin{aligned} 68. A &= \int_1^4 \frac{x+4}{x} dx = \int_1^4 \left(1 + \frac{4}{x}\right) dx \\ &= \left[x + 4 \ln x \right]_1^4 \\ &= 4 + 4 \ln 4 - 1 \\ &= 3 + 4 \ln 4 \approx 8.5452 \end{aligned}$$



$$\begin{aligned} 70. \int_1^4 (2x - \tan(0.3x)) dx &= \left[x^2 + \frac{10}{3} \ln |\cos(0.3x)| \right]_1^4 \\ &= \left[16 + \frac{10}{3} \ln \cos(1.2) \right] - \left[1 + \frac{10}{3} \ln \cos(0.3) \right] \approx 11.7686 \end{aligned}$$



72. Substitution: ($u = x^2 + 4$) and Power Rule

74. Substitution: ($u = \tan x$) and Log Rule

76. Answers will vary.

$$\begin{aligned} 78. \text{Average value} &= \frac{1}{4-2} \int_2^4 \frac{4(x+1)}{x^2} dx \\ &= 2 \int_2^4 \left(\frac{1}{x} + \frac{1}{x^2} \right) dx \\ &= 2 \left[\ln x - \frac{1}{x} \right]_2^4 \\ &= 2 \left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2} \right] \\ &= 2 \left[\ln 2 + \frac{1}{4} \right] = \ln 4 + \frac{1}{2} \approx 1.8863 \end{aligned}$$

$$\begin{aligned} 80. \text{Average value} &= \frac{1}{2-0} \int_0^2 \sec \frac{\pi x}{6} dx \\ &= \left[\frac{1}{2} \left(\frac{6}{\pi} \right) \ln \left| \sec \frac{\pi x}{6} + \tan \frac{\pi x}{6} \right| \right]_0^2 \\ &= \frac{3}{\pi} [\ln(2 + \sqrt{3}) - \ln(1 + 0)] \\ &= \frac{3}{\pi} \ln(2 + \sqrt{3}) \end{aligned}$$

$$\begin{aligned} 82. t &= \frac{10}{\ln 2} \int_{250}^{300} \frac{1}{T-100} dT \\ &= \frac{10}{\ln 2} \left[\ln(T-100) \right]_{250}^{300} = \frac{10}{\ln 2} [\ln 200 - \ln 150] = \frac{10}{\ln 2} \left[\ln \left(\frac{4}{3} \right) \right] \approx 4.1504 \text{ units of time} \end{aligned}$$

$$84. \frac{dS}{dt} = \frac{k}{t}$$

$$S(t) = \int \frac{k}{t} dt = k \ln |t| + C = k \ln t + C \text{ since } t > 1.$$

$$S(2) = k \ln 2 + C = 200$$

$$S(4) = k \ln 4 + C = 300$$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. Thus,

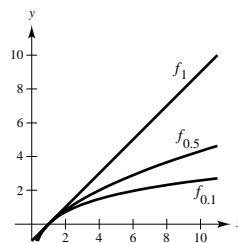
$$S(t) = \frac{100 \ln t}{\ln 2} + 100 = 100 \left[\frac{\ln t}{\ln 2} + 1 \right].$$

86. $k = 1: f_1(x) = x - 1$

$$k = 0.5: f_{0.5}(x) = \frac{\sqrt{x} - 1}{0.5} = 2(\sqrt{x} - 1)$$

$$k = 0.1: f_{0.1}(x) = \frac{\sqrt[10]{x} - 1}{0.1} = 10(\sqrt[10]{x} - 1)$$

$$\lim_{k \rightarrow 0^+} f_k(x) = \ln x$$



88. False

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

90. False; the integrand has a nonremovable discontinuity at $x = 0$.

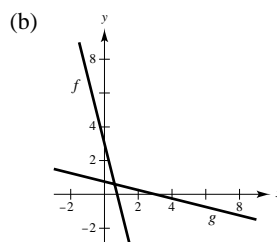
Section 5.3 Inverse Functions

2. (a) $f(x) = 3 - 4x$

$$g(x) = \frac{3 - x}{4}$$

$$f(g(x)) = f\left(\frac{3 - x}{4}\right) = 3 - 4\left(\frac{3 - x}{4}\right) = x$$

$$g(f(x)) = g(3 - 4x) = \frac{3 - (3 - 4x)}{4} = x$$

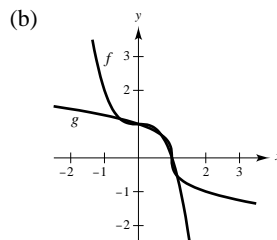


4. (a) $f(x) = 1 - x^3$

$$g(x) = \sqrt[3]{1 - x}$$

$$f(g(x)) = f(\sqrt[3]{1 - x}) = 1 - (\sqrt[3]{1 - x})^3 = 1 - (1 - x) = x$$

$$g(f(x)) = g(1 - x^3) = \sqrt[3]{1 - (1 - x^3)} = \sqrt[3]{x^3} = x$$

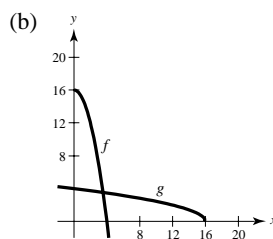


6. (a) $f(x) = 16 - x^2, x \geq 0$

$$g(x) = \sqrt{16 - x}$$

$$f(g(x)) = f(\sqrt{16 - x}) = 16 - (\sqrt{16 - x})^2 = 16 - (16 - x) = x$$

$$g(f(x)) = g(16 - x^2) = \sqrt{16 - (16 - x^2)} = \sqrt{x^2} = x$$

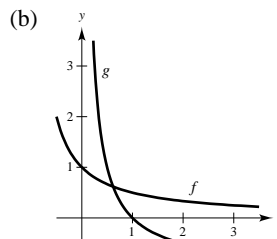


8. (a) $f(x) = \frac{1}{1 + x}, x \geq 0$

$$g(x) = \frac{1 - x}{x}, 0 < x \leq 1$$

$$f(g(x)) = f\left(\frac{1 - x}{x}\right) = \frac{1}{1 + \frac{1 - x}{x}} = \frac{1}{\frac{1 + x}{x}} = \frac{x}{1 + x} = x$$

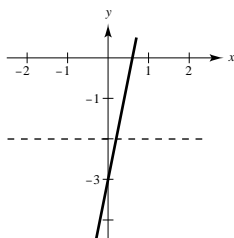
$$g(f(x)) = g\left(\frac{1}{1 + x}\right) = \frac{1 - \frac{1}{1 + x}}{\frac{1}{1 + x}} = \frac{\frac{1 + x - 1}{1 + x}}{\frac{1}{1 + x}} = \frac{x}{1 + x} \cdot \frac{1 + x}{1} = x$$



10. Matches (b)

14. $f(x) = 5x - 3$

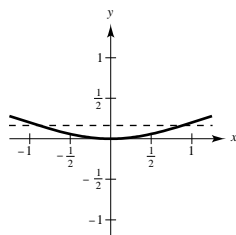
One-to-one; has an inverse



12. Matches (d)

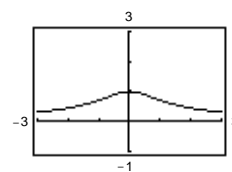
16. $F(x) = \frac{x^2}{x^2 + 4}$

Not one-to-one; does not have an inverse



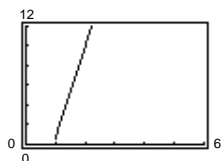
18. $g(t) = \frac{1}{\sqrt{t^2 + 1}}$

Not one-to-one; does not have an inverse



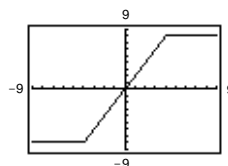
20. $f(x) = 5x\sqrt{x-1}$

One-to-one; has an inverse



22. $h(x) = |x + 4| - |x - 4|$

Not one-to-one; does not have an inverse



24. $f(x) = \cos \frac{3x}{2}$

$f'(x) = -\frac{3}{2} \sin \frac{3x}{2} = 0$ when $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \dots$

f is not strictly monotonic on $(-\infty, \infty)$. Therefore, f does not have an inverse.

26. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 = 0$ for all x .

f is increasing on $(-\infty, \infty)$. Therefore, f is strictly monotonic and has an inverse.

28. $f(x) = \ln(x - 3), x > 3$

$f'(x) = \frac{1}{x - 3} > 0$ for $x > 3$.

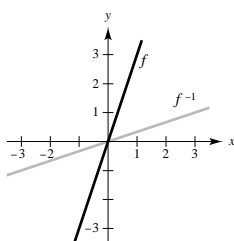
f is increasing on $(3, \infty)$. Therefore, f is strictly monotonic and has an inverse.

30. $f(x) = 3x = y$

$x = \frac{y}{3}$

$y = \frac{x}{3}$

$f^{-1}(x) = \frac{x}{3}$

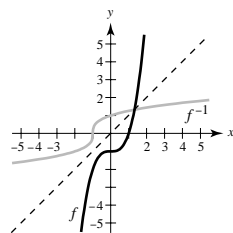


32. $f(x) = x^3 - 1 = y$

$x = \sqrt[3]{y + 1}$

$y = \sqrt[3]{x + 1}$

$f^{-1}(x) = \sqrt[3]{x + 1}$

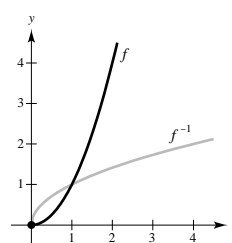


34. $f(x) = x^2 = y, 0 \leq x$

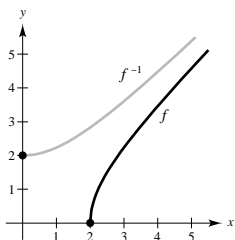
$x = \sqrt{y}$

$y = \sqrt{x}$

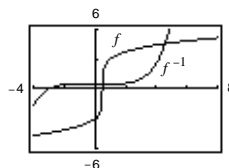
$f^{-1}(x) = \sqrt{x}$



$$\begin{aligned}
 36. \quad f(x) &= \sqrt{x^2 - 4} = y, \quad x \geq 2 \\
 x &= \sqrt{y^2 + 4} \\
 y &= \sqrt{x^2 - 4} \\
 f^{-1}(x) &= \sqrt{x^2 + 4}, \quad x \geq 0
 \end{aligned}$$

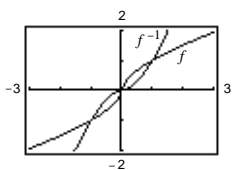


$$\begin{aligned}
 38. \quad f(x) &= 3\sqrt[5]{2x - 1} = y \\
 x &= \frac{y^5 + 243}{486} \\
 y &= \frac{x^5 + 243}{486} \\
 f^{-1}(x) &= \frac{x^5 + 243}{486}
 \end{aligned}$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

$$\begin{aligned}
 40. \quad f(x) &= x^{3/5} = y \\
 x &= y^{5/3} \\
 y &= x^{5/3} \\
 f^{-1}(x) &= x^{5/3}
 \end{aligned}$$



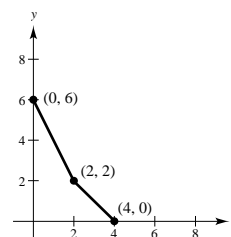
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

$$\begin{aligned}
 42. \quad f(x) &= \frac{x + 2}{x} = y \\
 x &= \frac{2}{y - 1} \\
 y &= \frac{2}{x - 1} \\
 f^{-1}(x) &= \frac{2}{x - 1}
 \end{aligned}$$



The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

x	0	2	4
$f^{-1}(x)$	6	2	0



$$46. \quad f(x) = k(2 - x - x^3) \text{ is one-to-one for all } k \neq 0. \text{ Since } f^{-1}(3) = -2, f(-2) = 3 = k(2 - (-2) - (-2)^3) = 12k \quad k = \frac{1}{4}.$$

$$48. \quad f(x) = |x + 2| \text{ on } [-2, \infty)$$

$$f'(x) = \frac{|x + 2|}{x + 2}(1) = 1 > 0 \text{ on } (-2, \infty)$$

f is increasing on $[-2, \infty)$. Therefore, f is strictly monotonic and has an inverse.

$$50. \quad f(x) = \cot x \text{ on } (0, \pi)$$

$$f'(x) = -\csc^2 x < 0 \text{ on } (0, \pi)$$

f is decreasing on $(0, \pi)$. Therefore, f is strictly monotonic and has an inverse.

$$52. \quad f(x) = \sec x \text{ on } \left[0, \frac{\pi}{2}\right)$$

$$f'(x) = \sec x \tan x > 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

f is increasing on $[0, \pi/2)$. Therefore, f is strictly monotonic and has an inverse.

54. $f(x) = 2 - \frac{3}{x^2} = y$ on $(0, 10)$

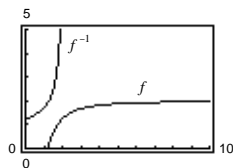
$$2x^2 - 3 = x^2y$$

$$x^2(2 - y) = 3$$

$$x = \pm \sqrt{\frac{3}{2 - y}}$$

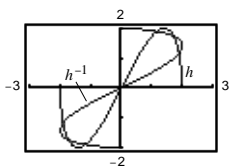
$$y = \pm \sqrt{\frac{3}{2 - x}}$$

$$f^{-1}(x) = \sqrt{\frac{3}{2 - x}}, x < 2$$



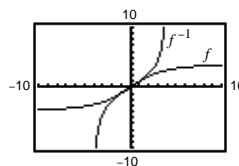
The graphs of f and f^{-1} are reflections of each other across the line $y = x$.

56. (a), (b)



(c) h is not one-to-one and does not have an inverse. The inverse relation is not an inverse function.

58. (a), (b)



(c) Yes, f is one-to-one and has an inverse. The inverse relation is an inverse function.

60. $f(x) = -3$

Not one-to-one; does not have an inverse

62. $f(x) = ax + b$

f is one-to-one; has an inverse

$$ax + b = y$$

$$x = \frac{y - b}{a}$$

$$y = \frac{x - b}{a}$$

$$f^{-1}(x) = \frac{x - b}{a}, a \neq 0$$

64. $f(x) = 16 - x^4$ is one-to-one for $x \geq 0$.

$$16 - x^4 = y$$

$$16 - y = x^4$$

$$\sqrt[4]{16 - y} = x$$

$$\sqrt[4]{16 - x} = y$$

$$f^{-1}(x) = \sqrt[4]{16 - x}, x \leq 16$$

66. $f(x) = |x - 3|$ is one-to-one for $x \geq 3$.

$$x - 3 = y$$

$$x = y + 3$$

$$y = x - 3$$

$$f^{-1}(x) = x - 3, x \geq 0$$

68. No, there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

70. Yes, the area function is increasing and hence one-to-one. The inverse function gives the radius r corresponding to the area A .

72. $f(x) = \frac{1}{27}(x^5 + 2x^3); f(-3) = \frac{1}{27}(-243 - 54) = -11 = a$.

$$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$$

$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))} = \frac{1}{f'(-3)} = \frac{27}{5(-3)^4 + 6(-3)^2} = \frac{1}{17}$$

74. $f(x) = \cos 2x, f(0) = 1 = a$

$$f'(x) = -2 \sin 2x$$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-2 \sin 0} = \frac{1}{0} \text{ which is undefined.}$$

76. $f(x) = \sqrt{x-4}, f(8) = 2 = a$

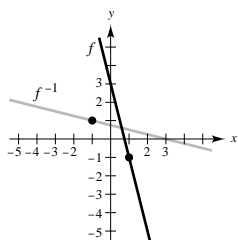
$$f'(x) = \frac{1}{2\sqrt{x-4}}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)} = \frac{1}{1/(2\sqrt{8-4})} = \frac{1}{1/4} = 4$$

78. (a) Domain $f = \text{Domain } f^{-1} = (-\infty, \infty)$

(b) Range $f = \text{Range } f^{-1} = (-\infty, \infty)$

(c)



(d) $f(x) = 3 - 4x, (1, -1)$

$$f'(x) = -4$$

$$f'(1) = -4$$

$$f^{-1}(x) = \frac{3-x}{4}, (-1, 1)$$

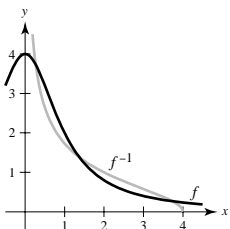
$$(f^{-1})'(x) = -\frac{1}{4}$$

$$(f^{-1})'(-1) = -\frac{1}{4}$$

80. (a) Domain $f = [0, \infty), \text{Domain } f^{-1} = (0, 4]$

(b) Range $f = (0, 4], \text{Range } f^{-1} = [0, \infty)$

(c)



(d) $f(x) = \frac{4}{1+x^2}$

$$f'(x) = \frac{-8x}{(x^2+1)^2}, f'(1) = -2$$

$$f^{-1}(x) = \sqrt{\frac{4-x}{x}}$$

$$(f^{-1})'(x) = \frac{-2}{x^2 \sqrt{\frac{4-x}{x}}}, (f^{-1})'(2) = -\frac{1}{2}$$

82. $x = 2 \ln(y^2 - 3)$

$$1 = 2 \frac{1}{y^2 - 3} 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^2 - 3}{4y}. \text{ At } (0, 4), \frac{dy}{dx} = \frac{16 - 3}{16} = \frac{13}{16}.$$

In Exercises 84 and 86, use the following.

$$f(x) = \frac{1}{8}x - 3 \text{ and } g(x) = x^3$$

$$f^{-1}(x) = 8(x+3) \text{ and } g^{-1}(x) = \sqrt[3]{x}$$

84. $(g^{-1} \circ f^{-1})(-3) = g^{-1}(f^{-1}(-3)) = g^{-1}(0) = 0$

86. $(g^{-1} \circ g^{-1})(-4) = g^{-1}(g^{-1}(-4)) = g^{-1}(\sqrt[3]{-4})$
 $= \sqrt[3]{\sqrt[3]{-4}} = -\sqrt[3]{4}$

In Exercises 88 and 90, use the following.

$$f(x) = x + 4 \text{ and } g(x) = 2x - 5$$

$$f^{-1}(x) = x - 4 \text{ and } g^{-1}(x) = \frac{x + 5}{2}$$

$$\begin{aligned} 88. (f^{-1} \circ g^{-1})(x) &= f^{-1}(g^{-1}(x)) \\ &= f^{-1}\left(\frac{x + 5}{2}\right) \\ &= \frac{x + 5}{2} - 4 \\ &= \frac{x - 3}{2} \end{aligned}$$

$$\begin{aligned} 90. (g \circ f)(x) &= g(f(x)) \\ &= g(x + 4) \\ &= 2(x + 4) - 5 \\ &= 2x + 3 \end{aligned}$$

$$\text{Hence, } (g \circ f)^{-1}(x) = \frac{x - 3}{2}$$

$$(\text{Note: } (g \circ f)^{-1} = f^{-1} \circ g^{-1})$$

92. The graphs of f and f^{-1} are mirror images with respect to the line $y = x$.

94. Theorem 5.9: Let f be differentiable on an interval I . If f has an inverse g , then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, f'(g(x)) \neq 0$$

96. f is not one-to-one because different x -values yield the same y -value.

$$\text{Example: } f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$$

Not continuous at ± 2 .

98. If f has an inverse, then f and f^{-1} are both one-to-one. Let $(f^{-1})^{-1}(x) = y$ then $x = f^{-1}(y)$ and $f(x) = y$. Thus, $(f^{-1})^{-1} = f$.

100. If f has an inverse and $f(x_1) = f(x_2)$, then $f^{-1}(f(x_1)) = f^{-1}(f(x_2))$ $x_1 = x_2$. Therefore, f is one-to-one. If $f(x)$ is one-to-one, then for every value b in the range, there corresponds exactly one value a in the domain. Define $g(x)$ such that the domain of g equals the range of f and $g(b) = a$. By the reflexive property of inverses, $g = f^{-1}$.

102. True; if f has a y -intercept.

104. False

$$\text{Let } f(x) = x \text{ or } g(x) = 1/x.$$

106. From Theorem 5.9, we have:

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ g''(x) &= \frac{f'(g(x))(0) - f''(g(x))g'(x)}{[f'(g(x))]^2} \\ &= -\frac{f''(g(x)) \cdot [1/f'(g(x))]}{[f'(g(x))]^2} \\ &= -\frac{f''(g(x))}{[f'(g(x))]^3} \end{aligned}$$

If f is increasing and concave down, then $f' > 0$ and $f'' < 0$ which implies that g is increasing and concave up.

Section 5.4 Exponential Functions: Differentiation and Integration

2. $e^{-2} = 0.1353\dots$

$\ln 0.1353\dots = -2$

4. $\ln 0.5 = -0.6931\dots$

$e^{-0.6931\dots} = \frac{1}{2}$

6. $e^{\ln 2x} = 12$

$2x = 12$

$x = 6$

8. $4e^x = 83$

$e^x = \frac{83}{4}$

$x = \ln\left(\frac{83}{4}\right) \approx 3.033$

10. $-6 + 3e^x = 8$

$3e^x = 14$

$e^x = \frac{14}{3}$

$x = \ln\left(\frac{14}{3}\right) \approx 1.540$

12. $200e^{-4x} = 15$

$e^{-4x} = \frac{15}{200} = \frac{3}{40}$

$-4x = \ln\left(\frac{3}{40}\right)$

$x = \frac{1}{4} \ln\left(\frac{40}{3}\right) \approx 0.648$

14. $\ln x^2 = 10$

$x^2 = e^{10}$

$x = \pm e^5 \approx \pm 148.4132$

16. $\ln 4x = 1$

$4x = e^1 = e$

$x = \frac{e}{4} \approx 0.680$

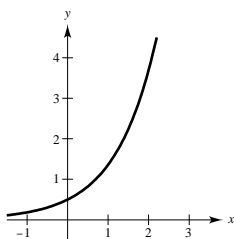
18. $\ln(x-2)^2 = 12$

$(x-2)^2 = e^{12}$

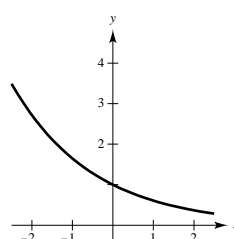
$x-2 = e^6$

$x = 2 + e^6 \approx 405.429$

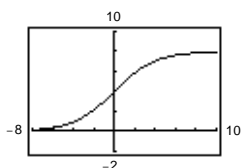
20. $y = \frac{1}{2}e^x$



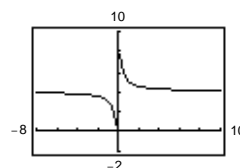
22. $y = e^{-x/2}$



24. (a)

Horizontal asymptotes: $y = 0$ and $y = 8$

(b)

Horizontal asymptote: $y = 4$

26. $y = Ce^{-ax}$

 Horizontal asymptote: $y = 0$

 Reflection in the y -axis

Matches (d)

28. $y = \frac{C}{1 + e^{-ax}}$

$$\lim_{x \rightarrow \infty} \frac{C}{1 + e^{-ax}} = C$$

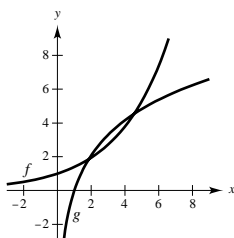
$$\lim_{x \rightarrow -\infty} \frac{C}{1 + e^{-ax}} = 0$$

 Horizontal asymptotes: $y = C$ and $y = 0$

Matches (b)

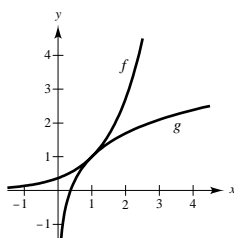
30. $f(x) = e^{x/3}$

$g(x) = \ln x^3 = 3 \ln x$



32. $f(x) = e^{x-1}$

$g(x) = 1 + \ln x$



34. In the same way,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x = e^r \text{ for } r > 0.$$

36. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} = 2.71825396$

$e \approx 2.718281828$

$e > 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

38. (a) $y = e^{2x}$

$y' = 2e^{2x}$

 At $(0, 1)$, $y' = 2$.

(b) $y = e^{-2x}$

$y' = -2e^{-2x}$

 At $(0, 1)$, $y' = -2$.

40. $f(x) = e^{1-x}$

$f'(x) = -e^{1-x}$

42. $y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

44. $y = x^2e^{-x}$

$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x}$

$= xe^{-x}(2 - x)$

46. $g(t) = e^{-3/t^2}$

$g'(t) = e^{-3/t^2}(6t^{-3}) = \frac{6}{t^3 e^{3/t^2}}$

48. $y = \ln\left(\frac{1+e^x}{1-e^x}\right)$

$= \ln(1+e^x) - \ln(1-e^x)$

$\frac{dy}{dx} = \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x}$

$= \frac{2e^x}{1-e^{2x}}$

50. $y = \ln\left(\frac{e^x + e^{-x}}{2}\right)$

$= \ln(e^x + e^{-x}) - \ln 2$

$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$= \frac{e^{2x} - 1}{e^{2x} + 1}$

52. $y = \frac{e^x - e^{-x}}{2}$

$\frac{dy}{dx} = \frac{e^x + e^{-x}}{2}$

54. $y = xe^x - e^x = e^x(x - 1)$

$\frac{dy}{dx} = e^x + e^x(x - 1) = xe^x$

56. $f(x) = e^3 \ln x$

$$f'(x) = \frac{e^3}{x}$$

58. $y = \ln e^x = x$

$$\frac{dy}{dx} = 1$$

60. $e^{xy} + x^2 - y^2 = 10$

$$\left(x \frac{dy}{dx} + y\right)e^{xy} + 2x - 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

62. $g(x) = \sqrt{x} + e^x \ln x$

$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x$$

$$\begin{aligned} g''(x) &= -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x \\ &= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x - 1)}{x^2} + e^x \ln x \end{aligned}$$

64. $y = e^x(3 \cos 2x - 4 \sin 2x)$

$$y' = e^x(-6 \sin 2x - 8 \cos 2x) + e^x(3 \cos 2x - 4 \sin 2x)$$

$$= e^x(-10 \sin 2x - 5 \cos 2x) = -5e^x(2 \sin 2x + \cos 2x)$$

$$y'' = -5e^x(4 \cos 2x - 2 \sin 2x) - 5e^x(2 \sin 2x + \cos 2x) = -5e^x(5 \cos 2x) = -25e^x \cos 2x$$

$$y'' - 2y' = -25e^x \cos 2x - 2(-5e^x)(2 \sin 2x + \cos 2x) = -5e^x(3 \cos 2x - 4 \sin 2x) = -5y$$

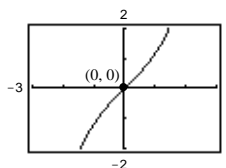
$$\text{Therefore, } y'' - 2y' = -5y \quad y'' - 2y' + 5y = 0.$$

66. $f(x) = \frac{e^x - e^{-x}}{2}$

$$f'(x) = \frac{e^x + e^{-x}}{2} > 0$$

$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$



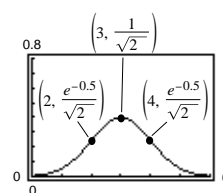
68. $g(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}$

$$g'(x) = \frac{-1}{\sqrt{2\pi}}(x-3)e^{-(x-3)^2/2}$$

$$g''(x) = \frac{1}{\sqrt{2\pi}}(x-2)(x-4)e^{-(x-3)^2/2}$$

$$\text{Relative maximum: } \left(3, \frac{1}{\sqrt{2\pi}}\right) \approx (3, 0.399)$$

$$\text{Points of inflection: } \left(2, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right), \left(4, \frac{1}{\sqrt{2\pi}}e^{-1/2}\right) \approx (2, 0.242), (4, 0.242)$$



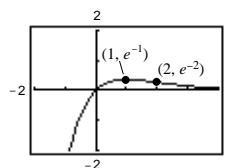
70. $f(x) = xe^{-x}$

$$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) = 0 \text{ when } x = 1.$$

$$f''(x) = -e^{-x} + (-e^{-x})(1-x) = e^{-x}(x-2) = 0 \text{ when } x = 2.$$

Relative maximum: $(1, e^{-1})$

Point of inflection: $(2, 2e^{-2})$



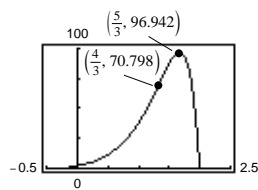
72. $f(x) = -2 + e^{3x}(4 - 2x)$

$$f'(x) = e^{3x}(-2) + 3e^{3x}(4 - 2x) = e^{3x}(10 - 6x) = 0 \text{ when } x = \frac{5}{3}.$$

$$f''(x) = e^{3x}(-6) + 3e^{3x}(10 - 6x) = e^{3x}(24 - 18x) = 0 \text{ when } x = \frac{4}{3}.$$

Relative maximum: $(\frac{5}{3}, 96.942)$

Point of inflection: $(\frac{4}{3}, 70.798)$



74. (a) $f(c) = f(c + x)$

$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$

$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$

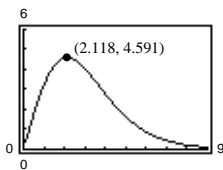
$$ce^{c+x} = (c + x)e^c$$

$$ce^x = c + x$$

$$ce^x - c = x$$

$$c = \frac{x}{e^x - 1}$$

(c) $A(x) = \frac{10x^2}{e^x - 1} e^{x/(1-e^x)}$

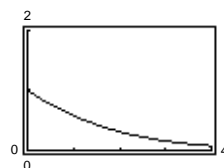


The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(b)
$$A(x) = xf(c) = x \left[10 \left(\frac{x}{e^x - 1} \right) e^{-(x/(e^x - 1))} \right]$$

$$= \frac{10x^2}{e^x - 1} e^{x/(1-e^x)}$$

(d)
$$c = \frac{x}{e^x - 1}$$



$$\lim_{x \rightarrow 0^+} c = 1$$

$$\lim_{x \rightarrow \infty} c = 0$$

76. Let (x_0, y_0) be the desired point on $y = e^{-x}$.

$$y = e^{-x}$$

$$y' = -e^{-x} \quad (\text{Slope of tangent line})$$

$$-\frac{1}{y'} = e^x \quad (\text{Slope of normal line})$$

$$y - e^{-x_0} = e^{x_0}(x - x_0)$$

We want $(0, 0)$ to satisfy the equation:

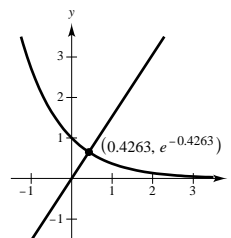
$$-e^{-x_0} = -x_0 e^{x_0}$$

$$1 = x_0 e^{2x_0}$$

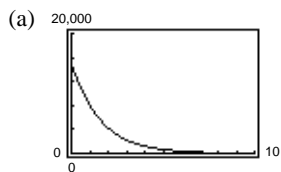
$$x_0 e^{2x_0} - 1 = 0$$

Solving by Newton's Method or using a computer, the solution is $x_0 \approx 0.4263$.

$$(0.4263, e^{-0.4263})$$



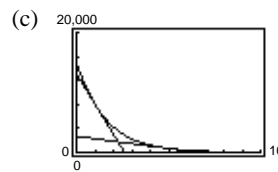
78. $V = 15,000e^{-0.6286t}, 0 \leq t \leq 10$



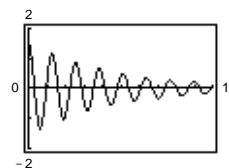
(b) $\frac{dV}{dt} = -9429e^{-0.6286t}$

When $t = 1$, $\frac{dV}{dt} \approx -5028.84$.

When $t = 5$, $\frac{dV}{dt} \approx -406.89$.

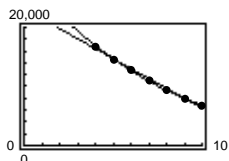


80. $1.56e^{-0.22t} \cos 4.9t - 0.25$ (3 inches equals one-fourth foot.) Using a graphing utility or Newton's Method, we have $t \approx 7.79$ seconds.



82. (a) $V_1 = -1686.79t + 23,181.79$

$V_2 = 109.52t^2 - 3220.12t + 28,110.36$



(b) The slope represents the rate of decrease in value of the car.

(c) $V_3 = 31,450.77(0.8592)^t = 31,450.77e^{-0.1518t}$

 (d) Horizontal asymptote: $\lim_{t \rightarrow \infty} V_3(t) = 0$

 As $t \rightarrow \infty$, the value of the car approaches 0.

(e) $\frac{dV_3}{dt} = -4774.2e^{-0.1518t}$

For $t = 5$, $\frac{dV_3}{dt} \approx -2235$ dollars/year.

For $t = 9$, $\frac{dV_3}{dt} \approx -1218$ dollars/year.

84. $f(x) = e^{-x^2/2}, f(0) = 1$

$f'(x) = -xe^{-x^2/2}, f'(0) = 0$

$f''(x) = x^2e^{-x^2/2} - e^{-x^2/2} = e^{-x^2/2}(x^2 - 1), f''(0) = -1$

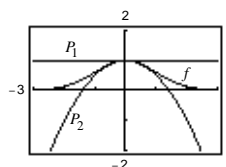
$P_1(x) = 1 + 0(x - 0) = 1, P_1(0) = 1$

$P_1'(x) = 0, P_1'(0) = 0$

$P_2(x) = 1 + 0(x - 0) - \frac{1}{2}(x - 0)^2 = 1 - \frac{x^2}{2}, P_2(0) = 1$

$P_2'(x) = -x, P_2'(0) = 0$

$P_2''(x) = -1, P_2''(0) = -1$

 The values of f, P_1, P_2 and their first derivatives agree at $x = 0$. The values of the second derivatives of f and P_2 agree at $x = 0$.


86. n^{th} term is $x^n/n!$ in polynomial:

$$y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Conjecture: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

90. $\int_3^4 e^{3-x} dx = \left[-e^{3-x} \right]_3^4 = -e^{-1} + 1 = 1 - \frac{1}{e}$

94. $\int \frac{e^{1/x^2}}{x^3} dx = -\frac{1}{2} \int e^{1/x^2} \left(\frac{-2}{x^3} \right) dx = -\frac{1}{2} e^{1/x^2} + C$

98. Let $u = \frac{-x^2}{2}$, $du = -x dx$.

$$\int_0^{\sqrt{2}} x e^{-x^2/2} dx = - \int_0^{\sqrt{2}} e^{-x^2/2} (-x) dx = \left[-e^{-x^2/2} \right]_0^{\sqrt{2}} = 1 - e^{-1} = \frac{e-1}{e}$$

100. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x})dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \ln(e^x + e^{-x}) + C$$

104. $\int \frac{e^{2x} + 2e^x + 1}{e^x} dx = \int (e^x + 2 + e^{-x}) dx$
 $= e^x + 2x - e^{-x} + C$

108. $\int \ln(e^{2x-1}) dx = \int (2x - 1) dx$
 $= x^2 - x + C$

112. $f'(x) = \int (\sin x + e^{2x}) dx = -\cos x + \frac{1}{2} e^{2x} + C_1$

$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \quad C_1 = 1$$

$$f'(x) = -\cos x + \frac{1}{2} e^{2x} + 1$$

$$f(x) = \int \left(-\cos x + \frac{1}{2} e^{2x} + 1 \right) dx$$

$$= -\sin x + \frac{1}{4} e^{2x} + x + C_2$$

$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \quad C_2 = 0$$

$$f(x) = x - \sin x + \frac{1}{4} e^{2x}$$

88. Let $u = -x^4$, $du = -4x^3 dx$.

$$\int e^{-x^4} (-4x^3) dx = e^{-x^4} + C$$

92. $\int x^2 e^{x^3/2} dx = \frac{2}{3} \int e^{x^3/2} \left(\frac{3x^2}{2} \right) dx = \frac{2}{3} e^{x^3/2} + C$

96. Let $u = 1 + e^{2x}$, $du = 2e^{2x} dx$.

$$\int \frac{e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \int \frac{2e^{2x}}{1 + e^{2x}} dx = \frac{1}{2} \ln(1 + e^{2x}) + C$$

102. Let $u = e^x + e^{-x}$, $du = (e^x - e^{-x}) dx$.

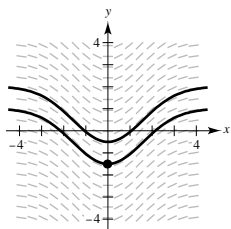
$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx = 2 \int (e^x + e^{-x})^{-2} (e^x - e^{-x}) dx$$

$$= \frac{-2}{e^x + e^{-x}} + C$$

106. $\int e^{\sec 2x} \sec 2x \tan 2x dx = \frac{1}{2} e^{\sec 2x} + C$
 $(u = \sec 2x, du = 2 \sec 2x \tan 2x)$

110. $y = \int (e^x - e^{-x})^2 dx$
 $= \int (e^{2x} - 2 + e^{-2x}) dx$
 $= \frac{1}{2} e^{2x} - 2x - \frac{1}{2} e^{-2x} + C$

114. (a)



(b) $\frac{dy}{dx} = xe^{-0.2x^2}, \left(0, -\frac{3}{2}\right)$

$$y = \int xe^{-0.2x^2} dx = \frac{1}{-0.4} \int e^{-0.2x^2} (-0.4x) dx$$

$$= -\frac{1}{0.4} e^{-0.2x^2} + C = -2.5e^{-0.2x^2} + C$$

$$\left(0, -\frac{3}{2}\right): -\frac{3}{2} = -2.5e^0 + C = -2.5 + C \quad C = 1$$

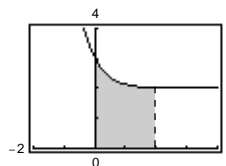
$$y = -2.5e^{-0.2x^2} + 1$$

116. $\int_a^b e^{-x} dx = \left[-e^{-x}\right]_a^b = e^{-a} - e^{-b}$



118. $\int_0^2 (e^{-2x} + 2) dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2$

$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491$$



120. (a) $\int_0^4 \sqrt{x}e^x dx, n = 12$

Midpoint Rule: 92.1898

Trapezoidal Rule: 93.8371

Simpson's Rule: 92.7385

Graphing Utility: 92.7437

(b) $\int_0^2 2xe^{-x} dx, n = 12$

Midpoint Rule: 1.1906

Trapezoidal Rule: 1.1827

Simpson's Rule: 1.1880

Graphing Utility: 1.18799

122. $\int_0^x 0.3^{-0.3t} dt = \frac{1}{2}$

$$\left[-e^{-0.3t}\right]_0^x = \frac{1}{2}$$

$$-e^{-0.3x} + 1 = \frac{1}{2}$$

$$e^{-0.3x} = \frac{1}{2}$$

$$-0.3x = \ln \frac{1}{2} = -\ln 2$$

$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes}$$

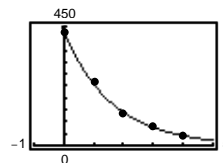
124.

t	0	1	2	3	4
R	425	240	118	71	36
$\ln R$	6.052	5.481	4.771	4.263	3.584

(a) $\ln R = -0.6155t + 6.0609$

$$R = e^{-0.6155t + 6.0609} = 428.78e^{-0.6155t}$$

(b)



(c) $\int_0^4 R(t) dt = \int_0^4 428.78e^{-0.6155t} dt \approx 637.2 \text{ liters}$

126. The graphs of $f(x) = \ln x$ and $g(x) = e^x$ are mirror images across the line $y = x$.

128. (a) Log Rule: ($u = e^x + 1$)

(b) Substitution: ($u = x^2$)

130. $\ln \frac{e^a}{e^b} = \ln e^a - \ln e^b = a - b$

$$\ln e^{a-b} = a - b$$

Therefore, $\ln \frac{e^a}{e^b} = \ln e^{a-b}$ and since $y = \ln x$ is one-to-one, we have $\frac{e^a}{e^b} = e^{a-b}$.

Section 5.5 Bases Other than e and Applications

2. $y = \left(\frac{1}{2}\right)^{t/8}$

At $t_0 = 16$, $y = \left(\frac{1}{2}\right)^{16/8} = \frac{1}{4}$

4. $y = \left(\frac{1}{2}\right)^{t/5}$

At $t_0 = 2$, $y = \left(\frac{1}{2}\right)^{2/5} \approx 0.7579$

6. $\log_{27} 9 = \log_{27} 27^{2/3} = \frac{2}{3}$

8. $\log_a \frac{1}{a} = \log_a 1 - \log_a a = -1$

10. (a) $27^{2/3} = 9$

$$\log_{27} 9 = \frac{2}{3}$$

(b) $16^{3/4} = 8$

$$\log_{16} 8 = \frac{3}{4}$$

12. (a) $\log_3 \frac{1}{9} = -2$

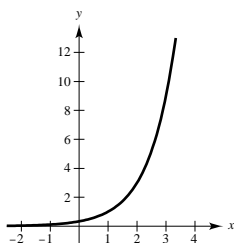
$$3^{-2} = \frac{1}{9}$$

(b) $49^{1/2} = 7$

$$\log_{49} 7 = \frac{1}{2}$$

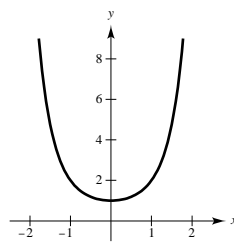
14. $y = 3^{x-1}$

x	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9



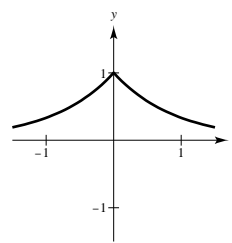
16. $y = 2^{x^2}$

x	-2	-1	0	1	2
y	16	2	1	2	16



18. $y = 3^{-|x|}$

x	0	± 1	± 2
y	1	$\frac{1}{3}$	$\frac{1}{9}$



20. (a) $\log_3 \frac{1}{81} = x$

$$3^x = \frac{1}{81}$$

$$x = -4$$

(b) $\log_6 36 = x$

$$6^x = 36$$

$$x = 2$$

22. (a) $\log_b 27 = 3$

$$b^3 = 27$$

$$b = 3$$

(b) $\log_b 125 = 3$

$$b^3 = 125$$

$$b = 5$$

24. (a) $\log_3 x + \log_3(x - 2) = 1$

$$\log_3[x(x - 2)] = 1$$

$$x(x - 2) = 3^1$$

$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$x = -1 \text{ OR } x = 3$$

$x = 3$ is the only solution since the domain of the logarithmic function is the set of all *positive* real numbers.

(b) $\log_{10}(x + 3) - \log_{10} x = 1$

$$\log_{10} \frac{x + 3}{x} = 1$$

$$\frac{x + 3}{x} = 10^1$$

$$x + 3 = 10x$$

$$3 = 9x$$

$$x = \frac{1}{3}$$

26. $5^{6x} = 8320$

$$6x \ln 5 = \ln 8320$$

$$x = \frac{\ln 8320}{6 \ln 5} \approx 0.935$$

28. $3(5^{x-1}) = 86$

$$5^{x-1} = \frac{86}{3}$$

$$(x - 1) \ln 5 = \ln\left(\frac{86}{3}\right)$$

$$x - 1 = \frac{\ln\left(\frac{86}{3}\right)}{\ln 5}$$

$$x = 1 + \frac{\ln\left(\frac{86}{3}\right)}{\ln 5} \approx 3.085$$

30. $\left(1 + \frac{0.10}{365}\right)^{365t} = 2$

$$365t \ln\left(1 + \frac{0.10}{365}\right) = \ln 2$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.10}{365}\right)} \approx 6.932$$

32. $\log_{10}(t - 3) = 2.6$

$$t - 3 = 10^{2.6}$$

$$t = 3 + 10^{2.6} \approx 401.107$$

34. $\log_5 \sqrt{x - 4} = 3.2$

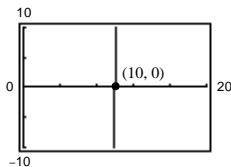
$$\sqrt{x - 4} = 5^{3.2}$$

$$x - 4 = (5^{3.2})^2 = 5^{6.4}$$

$$x = 4 + 5^{6.4} \approx 29,748.593$$

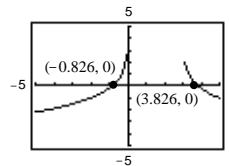
36. $f(t) = 300(1.0075^{12t}) - 735.41$

Zero: $t \approx 10$



38. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

Zeros: $x \approx -0.826, 3.826$

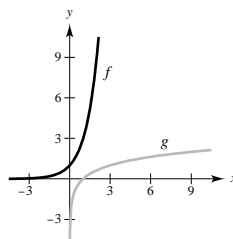


40. $f(x) = 3^x$

$g(x) = \log_3 x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)$	-2	-1	0	1	2



42. $g(x) = 2^{-x}$

$g'(x) = -(\ln 2) 2^{-x}$

44. $y = x(6^{-2x})$

$$\frac{dy}{dx} = x[-2(\ln 6)6^{-2x}] + 6^{-2x}$$

$$= 6^{-2x}[-2x(\ln 6) + 1]$$

$$= 6^{-2x}(1 - 2x \ln 6)$$

46. $f(t) = \frac{3^{2t}}{t}$

$$f'(t) = \frac{t(2 \ln 3) 3^{2t} - 3^{2t}}{t^2}$$

$$= \frac{3^{2t}(2t \ln 3 - 1)}{t^2}$$

48. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

$$g'(\alpha) = 5^{-\alpha/2} 2 \cos 2\alpha - \frac{1}{2}(\ln 5) 5^{-\alpha/2} \sin 2\alpha$$

50. $y = \log_{10}(2x) = \log_{10} 2 + \log_{10} x$

$$\frac{dy}{dx} = 0 + \frac{1}{x \ln 10} = \frac{1}{x \ln 10}$$

52. $h(x) = \log_3 \frac{x\sqrt{x-1}}{2}$

$$= \log_3 x + \frac{1}{2} \log_3(x-1) - \log_3 2$$

$$h'(x) = \frac{1}{x \ln 3} + \frac{1}{2} \cdot \frac{1}{(x-1) \ln 3} - 0$$

$$= \frac{1}{\ln 3} \left[\frac{1}{x} + \frac{1}{2(x-1)} \right]$$

$$= \frac{1}{\ln 3} \left[\frac{3x-2}{2x(x-1)} \right]$$

54. $y = \log_{10} \frac{x^2-1}{x}$

$$= \log_{10}(x^2-1) - \log_{10} x$$

$$\frac{dy}{dx} = \frac{2x}{(x^2-1) \ln 10} - \frac{1}{x \ln 10}$$

$$= \frac{1}{\ln 10} \left[\frac{2x}{x^2-1} - \frac{1}{x} \right]$$

$$= \frac{1}{\ln 10} \left[\frac{x^2+1}{x(x^2-1)} \right]$$

56. $f(t) = t^{3/2} \log_2 \sqrt{t+1} = t^{3/2} \frac{1}{2} \frac{\ln(t+1)}{\ln 2}$

$$f'(t) = \frac{1}{2 \ln 2} \left[t^{3/2} \frac{1}{t+1} + \frac{3}{2} t^{1/2} \ln(t+1) \right]$$

58. $y = x^{x-1}$

$$\ln y = (x-1)(\ln x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = (x-1) \left(\frac{1}{x} \right) + \ln x$$

$$\frac{dy}{dx} = y \left[\frac{x-1}{x} + \ln x \right]$$

$$= x^{x-2} (x-1 + x \ln x)$$

60. $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \frac{1}{x} \left(\frac{1}{1+x} \right) + \ln(1+x) \left(-\frac{1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{y}{x} \left[\frac{1}{x+1} - \frac{\ln(x+1)}{x} \right]$$

$$= \frac{(1+x)^{1/x}}{x} \left[\frac{1}{x+1} - \frac{\ln(x+1)}{x} \right]$$

62. $\int 5^{-x} dx = \frac{-5^{-x}}{\ln 5} + C$

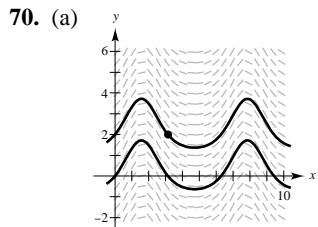
64. $\int_{-2}^0 (3^3 - 5^2) dx = \int_{-2}^0 (27 - 25) dx$

$$= \int_{-2}^0 2 dx$$

$$= \left[2x \right]_{-2}^0 = 4$$

$$\begin{aligned}
 66. \int (3-x) 7^{(3-x)^2} dx &= -\frac{1}{2} \int -2(3-x) 7^{(3-x)^2} dx \\
 &= -\frac{1}{2 \ln 7} [7^{(3-x)^2}] + C
 \end{aligned}$$

$$\begin{aligned}
 68. \int 2^{\sin x} \cos x dx, u &= \sin x, du = \cos x dx \\
 &= \frac{1}{\ln 2} 2^{\sin x} + C
 \end{aligned}$$



(b) $\frac{dy}{dx} = e^{\sin x} \cos x \quad (\pi, 2)$

$$y = \int e^{\sin x} \cos x dx = e^{\sin x} + C$$

$$(\pi, 2): 2 = e^{\sin \pi} + C = 1 + C \quad C = 1$$

$$y = e^{\sin x} + 1$$

72. $\log_b x = \frac{\ln x}{\ln b} = \frac{\log_{10} x}{\log_{10} b}$

74. $f(x) = \log_{10} x$

(a) Domain: $x > 0$

(b) $y = \log_{10} x$

$$10^y = x$$

$$f^{-1}(x) = 10^x$$

(c) $\log_{10} 1000 = \log_{10} 10^3 = 3$

$$\log_{10} 10,000 = \log_{10} 10^4 = 4$$

If 1000 x 10,000, then 3 $f(x)$ 4.

(d) If $f(x) < 0$, then $0 < x < 1$.

(e) $f(x) + 1 = \log_{10} x + \log_{10} 10$

$$= \log_{10}(10x)$$

x must have been increased by a factor of 10.

(f) $\log_{10}\left(\frac{x_1}{x_2}\right) = \log_{10} x_1 - \log_{10} x_2$

$$= 3n - n = 2n$$

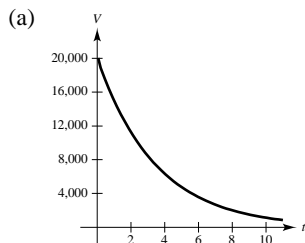
Thus, $x_1/x_2 = 10^{2n} = 100^n$.

76. $f(x) = a^x$

(a) $f(u+v) = a^{u+v} = a^u a^v = f(u)f(v)$

(b) $f(2x) = a^{2x} = (a^x)^2 = [f(x)]^2$

78. $V(t) = 20,000\left(\frac{3}{4}\right)^t$

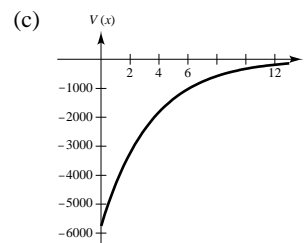


$$V(2) = 20,000\left(\frac{3}{4}\right)^2 = \$11,250$$

(b) $\frac{dV}{dt} = 20,000\left(\ln \frac{3}{4}\right)\left(\frac{3}{4}\right)^t$

When $t = 1$: $\frac{dV}{dt} \approx -4315.23$

When $t = 4$: $\frac{dV}{dt} \approx -1820.49$



Horizontal asymptote: $v' = 0$

As the car ages, it is worth less each year and depreciates less each year, but the value of the car will never reach \$0.

80. $P = \$2500$, $r = 6\% = 0.06$, $t = 20$

$$A = 2500 \left(1 + \frac{0.06}{n} \right)^{20n}$$

$$A = 2500e^{(0.06)(20)} = 8300.29$$

n	1	2	4	12	365	Continuous
A	8017.84	8155.09	8226.66	8275.51	8299.47	8300.29

82. $P = \$5000$, $r = 7\% = 0.07$, $t = 25$

$$A = 5000 \left(1 + \frac{0.07}{n} \right)^{25n}$$

$$A = 5000e^{0.07(25)}$$

n	1	2	4	12	365	Continuous
A	27,137.16	27,924.63	28,340.78	28,627.09	28,768.19	28,773.01

84. $100,000 = Pe^{0.06t}$ $P = 100,000e^{-0.06t}$

t	1	10	20	30	40	50
P	94,176.45	54,881.16	30,119.42	16,529.89	9071.80	4978.71

86. $100,000 = P \left(1 + \frac{0.07}{365} \right)^{365t}$ $P = 100,000 \left(1 + \frac{0.07}{365} \right)^{-365t}$

t	1	10	20	30	40	50
P	93,240.01	49,661.86	24,663.01	12,248.11	6082.64	3020.75

88. Let $P = \$100$, $0 < t < 20$.

(a) $A = 100e^{0.03t}$

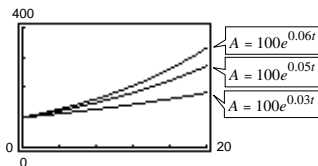
$$A(20) \approx 182.21$$

(b) $A = 100e^{0.05t}$

$$A(20) \approx 271.83$$

(c) $A = 100e^{0.06t}$

$$A(20) \approx 332.01$$



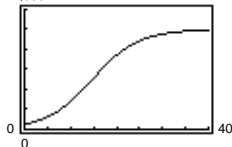
90. (a) $\lim_{n \rightarrow \infty} \frac{0.86}{1 + e^{-0.25n}} = 0.86$ or 86%

(b)
$$P' = \frac{-0.86(-0.25)(e^{-0.25n})}{(1 + e^{-0.25n})^2} = \frac{0.215e^{-0.25n}}{(1 + e^{-0.25n})^2}$$

$$P'(3) \approx 0.069$$

$$P'(10) \approx 0.016$$

92. (a)



(b) Limiting size: 10,000 fish

(c)
$$p(t) = \frac{10,000}{1 + 19e^{-t/5}}$$

$$p'(t) = \frac{e^{-t/5}}{(1 + 19e^{-t/5})^2} \left(\frac{19}{5} \right) (10,000)$$

$$= \frac{38,000e^{-t/5}}{(1 + 19e^{-t/5})^2}$$

$$p'(1) \approx 113.5 \text{ fish/month}$$

$$p'(10) \approx 403.2 \text{ fish/month}$$

(d)
$$p''(t) = -\frac{38,000}{5}(e^{-t/5}) \left[\frac{1 - 19e^{-t/5}}{(1 + 19e^{-t/5})^3} \right] = 0$$

$$19e^{-t/5} = 1$$

$$\frac{t}{5} = \ln 19$$

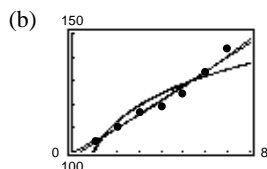
$$t = 5 \ln 19 \approx 14.72$$

94. (a) $y_1 = 6.0536x + 97.5571$

$$y_2 = 100.0751 + 17.8148 \ln x$$

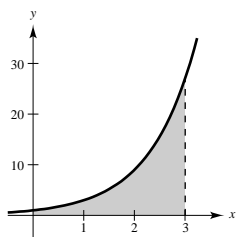
$$y_3 = 99.4557(1.0506)^x$$

$$y_4 = 101.2875x^{0.1471}$$



y_3 seems best.

96.
$$A = \int_0^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_0^3 = \frac{26}{\ln 3} \approx 23.666$$


 100.

t	0	1	2	3	4
y	600	630	661.50	694.58	729.30

$$y = C(k^t)$$

When $t = 0$, $y = 600$ $C = 600$.

$$y = 600(k^t)$$

$$\frac{630}{600} = 1.05, \quad \frac{661.50}{630} = 1.05, \quad \frac{694.58}{661.50} \approx 1.05, \quad \frac{729.30}{694.58} \approx 1.05$$

Let $k = 1.05$.

$$y = 600(1.05)^t$$

102. True.

$$\begin{aligned} f(e^{n+1}) - f(e^n) &= \ln e^{n+1} - \ln e^n \\ &= n + 1 - n \\ &= 1 \end{aligned}$$

104. True.

$$\begin{aligned} \frac{d^n y}{dx^n} &= Ce^x \\ &= y \text{ for } n = 1, 2, 3, \dots \end{aligned}$$

106. True.

$$\begin{aligned} f(x) &= g(x)e^x = 0 \\ g(x) &= 0 \text{ since } e^x > 0 \text{ for all } x. \end{aligned}$$

(c) The slope of 6.0536 is the annual rate of change in the amount given to philanthropy.

 (d) For 1996, $x = 6$ and $y_1' = 6.0536$, $y_2' \approx 2.9691$,
 $y_3' \approx 6.6015$, $y_4' \approx 3.2321$.

y_3 is increasing at the greatest rate in 1996.

 98.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1+x)^{1/x}$	2	2.594	2.705	2.718	2.718

108. $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \sin x \left(\frac{1}{x} \right) + \cos x \cdot \ln x$$

$$y' = x^{\sin x} \left[\frac{\sin x}{x} + \cos x \cdot \ln x \right]$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2} \right)$,

$$\begin{aligned} y' &= \left(\frac{\pi}{2} \right)^{\sin(\pi/2)} \left[\frac{\sin(\pi/2)}{\pi/2} + \cos\left(\frac{\pi}{2}\right) \ln\left(\frac{\pi}{2}\right) \right] \\ &= \frac{\pi}{2} \left[\frac{2}{\pi} + 0 \right] = 1 \end{aligned}$$

$$\begin{aligned} \text{Tangent line: } y - \frac{\pi}{2} &= 1 \left(x - \frac{\pi}{2} \right) \\ y &= x \end{aligned}$$

Section 5.6 Differential Equations: Growth and Decay

2. $\frac{dy}{dx} = 4 - x$

$$y = \int (4 - x) dx = 4x - \frac{x^2}{2} + C$$

4. $\frac{dy}{dx} = 4 - y$

$$\frac{dy}{4 - y} = dx$$

$$\int \frac{-1}{4 - y} dy = \int -dx$$

$$\ln|4 - y| dy = -x + C_1$$

$$4 - y = e^{-x+C_1} = Ce^{-x}$$

$$y = 4 - Ce^{-x}$$

6. $y' = \frac{\sqrt{x}}{3y}$

$$3yy' = \sqrt{x}$$

$$\int 3yy' dx = \int \sqrt{x} dx$$

$$\frac{3y^2}{2} = \frac{2}{3}x^{3/2} + C_1$$

$$9y^2 - 4x^{3/2} = C$$

8. $y' = x(1 + y)$

$$\frac{y'}{1 + y} = x$$

$$\int \frac{y'}{1 + y} dx = \int x dx$$

$$\int \frac{dy}{1 + y} = \int x dx$$

$$\ln(1 + y) = \frac{x^2}{2} + C_1$$

$$1 + y = e^{(x^2/2)+C_1}$$

$$y = e^{C_1} e^{x^2/2} - 1$$

$$= Ce^{x^2/2} - 1$$

10. $xy + y' = 100x$

$$y' = 100x + xy = x(100 - y)$$

$$\frac{y'}{100 - y} = x$$

$$\int \frac{y'}{100 - y} dx = \int x dx$$

$$\int \frac{1}{100 - y} dy = \int x dx$$

$$-\ln(100 - y) = \frac{x^2}{2} + C_1$$

$$\ln(100 - y) = -\frac{x^2}{2} - C_1$$

$$100 - y = e^{-(x^2/2)-C_1}$$

$$-y = e^{-C_1} e^{-x^2/2} - 100$$

$$y = 100 - Ce^{-x^2/2}$$

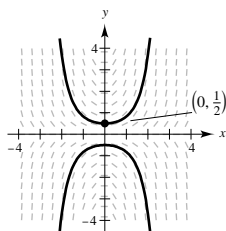
12. $\frac{dP}{dt} = k(10 - t)$

$$\int \frac{dP}{dt} dt = \int k(10 - t) dt$$

$$\int dP = -\frac{k}{2}(10 - t)^2 + C$$

$$P = -\frac{k}{2}(10 - t)^2 + C$$

16. (a)



14. $\frac{dy}{dx} = kx(L - y)$

$$\frac{1}{L - y} \frac{dy}{dx} = kx$$

$$\int \frac{1}{L - y} \frac{dy}{dx} dx = \int kx dx$$

$$\int \frac{1}{L - y} dy = \frac{kx^2}{2} + C_1$$

$$-\ln(L - y) = \frac{kx^2}{2} + C_1$$

$$L - y = e^{-(kx^2/2) - C_1}$$

$$-y = -L + e^{-C_1} e^{-kx^2/2}$$

$$y = L - Ce^{-kx^2/2}$$

(b) $\frac{dy}{dx} = xy, \left(0, \frac{1}{2}\right)$

$$\frac{dy}{y} = x dx$$

$$\ln|y| = \frac{x^2}{2} + C$$

$$y = e^{x^2/2 + C} = C_1 e^{x^2/2}$$

$$\left(0, \frac{1}{2}\right): \frac{1}{2} = C_1 e^0 \quad C_1 = \frac{1}{2} \quad y = \frac{1}{2} e^{x^2/2}$$

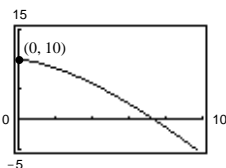
18. $\frac{dy}{dt} = -\frac{3}{4}\sqrt{t}, (0, 10)$

$$\int dy = \int -\frac{3}{4}\sqrt{t} dt$$

$$y = -\frac{1}{2}t^{3/2} + C$$

$$10 = -\frac{1}{2}(0)^{3/2} + C \quad C = 10$$

$$y = -\frac{1}{2}t^{3/2} + 10$$



20. $\frac{dy}{dt} = \frac{3}{4}y, (0, 10)$

$$\int \frac{dy}{y} = \int \frac{3}{4} dt$$

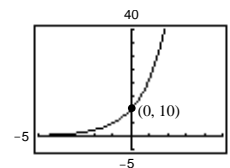
$$\ln y = \frac{3}{4}t + C_1$$

$$y = e^{(3/4)t + C_1}$$

$$= e^{C_1} e^{(3/4)t} = Ce^{3t/4}$$

$$10 = Ce^0 \quad C = 10$$

$$y = 10e^{3t/4}$$



22. $\frac{dN}{dt} = kN$

$$N = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 250): C = 250$$

$$(1, 400): 400 = 250e^k \quad k = \ln \frac{400}{250} = \ln \frac{8}{5}$$

$$\text{When } t = 4, N = 250e^{4 \ln(8/5)} = 250e^{\ln(8/5)^4}$$

$$= 250 \left(\frac{8}{5}\right)^4 = \frac{8192}{5}$$

24. $\frac{dP}{dt} = kP$

$$P = Ce^{kt} \quad (\text{Theorem 5.16})$$

$$(0, 5000): C = 5000$$

$$(1, 4750): 4750 = 5000e^k \quad k = \ln \left(\frac{19}{20}\right)$$

$$\text{When } t = 5, P = 5000e^{\ln(19/20)(5)}$$

$$= 5000 \left(\frac{19}{20}\right)^5 \approx 3868.905$$

$$26. y = Ce^{kt}, (0, 4), \left(5, \frac{1}{2}\right)$$

$$C = 4$$

$$y = 4e^{kt}$$

$$\frac{1}{2} = 4e^{5k}$$

$$k = \frac{\ln(1/8)}{5} \approx -0.4159$$

$$y = 4e^{-0.4159t}$$

$$28. y = Ce^{kt}, \left(3, \frac{1}{2}\right), (4, 5)$$

$$\frac{1}{2} = Ce^{3k}$$

$$5 = Ce^{4k}$$

$$2Ce^{3k} = \frac{1}{5}Ce^{4k}$$

$$10e^{3k} = e^{4k}$$

$$10 = e^k$$

$$k = \ln 10 \approx 2.3026$$

$$y = Ce^{2.3026t}$$

$$5 = Ce^{2.3026(4)}$$

$$C \approx 0.0005$$

$$y = 0.0005e^{2.3026t}$$

$$30. y' = \frac{dy}{dt} = ky$$

$$32. \frac{dy}{dx} = \frac{1}{2}x^2y$$

$$\frac{dy}{dx} > 0 \text{ when } y > 0. \text{ Quadrants I and II.}$$

34. Since $y = Ce^{\ln(1/2)/1620}t$, we have $1.5 = Ce^{\ln(1/2)/1620}(1000)$ $C \approx 2.30$ which implies that the initial quantity is 2.30 grams. When $t = 10,000$, we have $y = 2.30e^{\ln(1/2)/1620}(10,000) \approx 0.03$ gram.

36. Since $y = Ce^{\ln(1/2)/5730}t$, we have $2.0 = Ce^{\ln(1/2)/5730}(10,000)$ $C \approx 6.70$ which implies that the initial quantity is 6.70 grams. When $t = 1000$, we have $y = 6.70e^{\ln(1/2)/5730}(1000) \approx 5.94$ grams.

38. Since $y = Ce^{\ln(1/2)/5730}t$, we have $3.2 = Ce^{\ln(1/2)/5730}(1000)$ $C \approx 3.61$.

Initial quantity: 3.61 grams.

When $t = 10,000$, we have $y \approx 1.08$ grams.

40. Since $y = Ce^{\ln(1/2)/24,360}t$, we have $0.4 = Ce^{\ln(1/2)/24,360}(10,000)$ $C \approx 0.53$ which implies that the initial quantity is 0.53 gram. When $t = 1000$, we have $y = 0.53e^{\ln(1/2)/24,360}(1000) \approx 0.52$ gram.

42. Since $\frac{dy}{dx} = ky$, $y = Ce^{kt}$ or $y = y_0e^{kt}$.

$$\frac{1}{2}y_0 = y_0e^{5730k}$$

$$k = -\frac{\ln 2}{5730}$$

$$0.15y_0 = y_0e^{(-\ln 2/5730)t}$$

$$\ln 0.15 = -\frac{(\ln 2)t}{5730}$$

$$t = -\frac{5730 \ln 0.15}{\ln 2} \approx 15,682.8 \text{ years.}$$

44. Since $A = 20,000e^{0.055t}$, the time to double is given by $40,000 = 20,000e^{0.055t}$ and we have

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.6 \text{ years.}$$

Amount after 10 years:

$$A = 20,000e^{(0.055)(10)} \approx \$34,665.06$$

46. Since $A = 10,000e^{rt}$ and $A = 20,000$ when $t = 5$, we have the following.

$$20,000 = 10,000e^{5r}$$

$$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$$

Amount after 10 years: $A = 10,000e^{[(\ln 2)/5](10)} = \$40,000$

50. $500,000 = P\left(1 + \frac{0.06}{12}\right)^{(12)(40)}$

$$P = 500,000(1.005)^{-480} \approx \$45,631.04$$

54. (a) $2000 = 1000(1 + 0.6)^t$

$$2 = 1.06^t$$

$$\ln 2 = t \ln 1.06$$

$$t = \frac{\ln 2}{\ln 1.06} \approx 11.90 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.06}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.06}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.06}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.06}{12}\right)} \approx 11.58 \text{ years}$$

56. (a) $2000 = 1000(1 + 0.055)^t$

$$2 = 1.055^t$$

$$\ln 2 = t \ln 1.055$$

$$t = \frac{\ln 2}{\ln 1.055} \approx 12.95 \text{ years}$$

(b) $2000 = 1000\left(1 + \frac{0.055}{12}\right)^{12t}$

$$2 = \left(1 + \frac{0.055}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.055}{12}\right)$$

$$t = \frac{1}{12} \frac{\ln 2}{\ln\left(1 + \frac{0.055}{12}\right)} \approx 12.63 \text{ years}$$

48. Since $A = 2000e^{rt}$ and $A = 5436.56$ when $t = 10$, we have the following.

$$5436.56 = 2000e^{10r}$$

$$r = \frac{\ln(5436.56/2000)}{10} \approx 0.10 = 10\%$$

The time to double is given by

$$4000 = 2000e^{0.10t}$$

$$t = \frac{\ln 2}{0.10} \approx 6.93 \text{ years.}$$

52. $500,000 = P\left(1 + \frac{0.09}{12}\right)^{(12)(25)}$

$$P = 500,000\left(1 + \frac{0.09}{12}\right)^{-300}$$

$$\approx \$53,143.92$$

(c) $2000 = 1000\left(1 + \frac{0.06}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.06}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.06}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.06}{365}\right)} \approx 11.55 \text{ years}$$

(d) $2000 = 1000e^{0.06t}$

$$2 = e^{0.06t}$$

$$\ln 2 = 0.06t$$

$$t = \frac{\ln 2}{0.06} \approx 11.55 \text{ years}$$

(c) $2000 = 1000\left(1 + \frac{0.055}{365}\right)^{365t}$

$$2 = \left(1 + \frac{0.055}{365}\right)^{365t}$$

$$\ln 2 = 365t \ln\left(1 + \frac{0.055}{365}\right)$$

$$t = \frac{1}{365} \frac{\ln 2}{\ln\left(1 + \frac{0.055}{365}\right)} \approx 12.60 \text{ years}$$

(d) $2000 = 1000e^{0.055t}$

$$2 = e^{0.055t}$$

$$\ln 2 = 0.055t$$

$$t = \frac{\ln 2}{0.055} \approx 12.60 \text{ years}$$

58. $P = Ce^{kt} = Ce^{0.031t}$

$$P(-1) = 11.6 = Ce^{0.031(-1)} \quad C = 11.9652$$

$$P = 11.9652e^{0.031t}$$

$$P(10) \approx 16.31 \quad \text{or} \quad 16,310,000 \text{ people in 2010}$$

60. $P = Ce^{kt} = Ce^{-0.004t}$

$$P(-1) = 3.6 = Ce^{-0.004(-1)} \quad C = 3.5856$$

$$P = 3.5856e^{-0.004t}$$

$$P(10) \approx 3.45 \quad \text{or} \quad 3,450,000 \text{ people in 2010}$$

62. (a) $N = 100.1596(1.2455)^t$

(b) $N = 400$ when $t = 6.3$ hours (graphing utility)

Analytically,

$$400 = 100.1596(1.2455)^t$$

$$1.2455^t = \frac{400}{100.1596} = 3.9936$$

$$t \ln 1.2455 = \ln 3.9936$$

$$t = \frac{\ln 3.9936}{\ln 1.2455} \approx 6.3 \text{ hours.}$$

64. $y = Ce^{kt}$, $(0, 742,000)$, $(2, 632,000)$

$$C = 742,000$$

$$632,000 = 742,000e^{2k}$$

$$k = \frac{\ln(632/742)}{2} \approx -0.0802$$

$$y \approx 742,000e^{-0.0802t}$$

When $t = 4$, $y \approx \$538,372$.

66. (a) $20 = 30(1 - e^{30k})$

$$30e^{30k} = 10$$

$$k = \frac{\ln(1/3)}{30} = \frac{-\ln 3}{30} \approx -0.0366$$

$$N \approx 30(1 - e^{-0.0366t})$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$e^{-0.0366t} = \frac{1}{6}$$

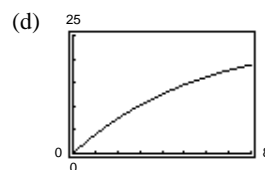
$$t = \frac{-\ln 6}{-0.0366} \approx 49 \text{ days}$$

68. $S = 25(1 - e^{kt})$

(a) $4 = 25(1 - e^{k(1)}) \quad 1 - e^k = \frac{4}{25} \quad e^k = \frac{21}{25} \quad k = \ln\left(\frac{21}{25}\right) \approx -0.1744$

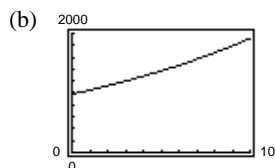
(b) 25,000 units $\left(\lim_{t \rightarrow \infty} S = 25\right)$

(c) When $t = 5$, $S \approx 14.545$ which is 14,545 units.

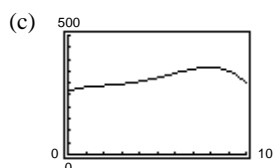


70. (a) $R = 979.3993(1.0694)^t = 979.3993e^{0.0671t}$

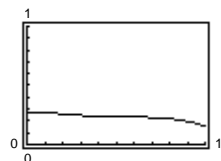
$$I = -0.1385t^4 + 2.1770t^3 - 9.9755t^2 + 23.8513t + 266.4923$$



Rate of growth = $R'(t) = 65.7e^{0.0671t}$



(d) $P(t) = \frac{I}{R}$



$$72. \quad 93 = 10 \log_{10} \frac{I}{10^{-16}} = 10(\log_{10} I + 16)$$

$$-6.7 = \log_{10} I \quad I = 10^{-6.7}$$

$$80 = 10 \log_{10} \frac{I}{10^{-16}} = 10(\log_{10} I + 16)$$

$$-8 = \log_{10} I \quad I = 10^{-8}$$

$$\text{Percentage decrease: } \left(\frac{10^{-6.7} - 10^{-8}}{10^{-6.7}} \right) (100) \approx 95\%$$

$$74. \quad \text{Since } \frac{dy}{dt} = k(y - 80)$$

$$\int \frac{1}{y - 80} dy = \int k dt$$

$$\ln(y - 80) = kt + C.$$

When $t = 0$, $y = 1500$. Thus, $C = \ln 1420$.

When $t = 1$, $y = 1120$. Thus,

$$k(1) + \ln 1420 = \ln(1120 - 80)$$

$$k = \ln 1040 - \ln 1420 = \ln \frac{104}{142}$$

Thus, $y = 1420e^{[\ln(104/142)]t} + 80$.

When $t = 5$, $y \approx 379.2^\circ$.

76. True

78. True

Section 5.7 Differential Equations: Separation of Variables

$$2. \text{ Differential equation: } y' = \frac{2xy}{x^2 - y^2}$$

$$\text{Solution: } x^2 + y^2 = Cy$$

$$\text{Check: } 2x + 2yy' = Cy'$$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy} = \frac{-2xy}{2y^2 - (x^2 + y^2)} = \frac{-2xy}{y^2 - x^2} = \frac{2xy}{x^2 - y^2}$$

$$4. \text{ Differential equation: } y'' + 2y' + 2y = 0$$

$$\text{Solution: } y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$$

$$\text{Check: } y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x +$$

$$2(-C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x + 2(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

$$6. \quad y = \frac{2}{3}(e^{-2x} + e^x)$$

$$y' = \frac{2}{3}(-2e^{-2x} + e^x)$$

$$y'' = \frac{2}{3}(4e^{-2x} + e^x)$$

$$\text{Substituting, } y'' + 2y' = \frac{2}{3}(4e^{-2x} + e^x) + 2\left(\frac{2}{3}\right)(-2e^{-2x} + e^x) = 2e^x.$$

In Exercises 8–12, the differential equation is $y^{(4)} - 16y = 0$.

8. $y = 3 \cos 2x$
 $y^{(4)} = 48 \cos 2x$
 $y^{(4)} - 16y = 48 \cos 2x - 48 \cos 2x = 0$, Yes.

10. $y = 5 \ln x$
 $y^{(4)} = -\frac{30}{x^4}$
 $y^{(4)} - 16y = -\frac{30}{x^4} - 80 \ln x \neq 0$, No.

12. $y = 3e^{2x} - 4 \sin 2x$
 $y^{(4)} = 48e^{2x} - 64 \sin 2x$
 $y^{(4)} - 16y = (48e^{2x} - 64 \sin 2x) - 16(3e^{2x} - 4 \sin 2x) = 0$, Yes

In 14–18, the differential equation is $xy' - 2y = x^3e^x$.

14. $y = x^2e^x$, $y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$
 $xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x$, Yes.

16. $y = \sin x$, $y' = \cos x$
 $xy' - 2y = x(\cos x) - 2(\sin x) \neq x^3e^x$, No.

18. $y = x^2e^x - 5x^2$, $y' = x^2e^x + 2xe^x - 10x$
 $xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2] = x^3e^x$, Yes.

20. $y = A \sin \omega t$
 $\frac{d^2y}{dt^2} = -A\omega^2 \sin \omega t$

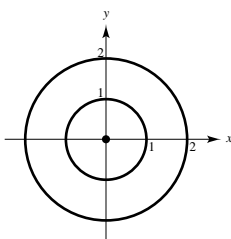
Since $(d^2y/dt^2) + 16y = 0$, we have

$$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0.$$

Thus, $\omega^2 = 16$ and $\omega = \pm 4$.

22. $2x^2 - y^2 = C$ passes through (3, 4)
 $2(9) - 16 = C \quad C = 2$
 Particular solution: $2x^2 - y^2 = 2$

24. Differential equation: $yy' + x = 0$
 General solution: $x^2 + y^2 = C$
 Particular solutions: $C = 0$, Point
 $C = 1$, $C = 4$, Circles



26. Differential equation: $3x + 2yy' = 0$
 General solution: $3x^2 + 2y^2 = C$
 $6x + 4yy' = 0$
 $2(3x + 2yy') = 0$
 $3x + 2yy' = 0$
 Initial condition:
 $y(1) = 3: 3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$
 Particular solution: $3x^2 + 2y^2 = 21$

28. Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$$y' = C_2 \left(\frac{1}{x}\right), y'' = -C_2 \left(\frac{1}{x^2}\right)$$

$$xy'' + y' = x \left(-C_2 \frac{1}{x^2}\right) + C_2 \frac{1}{x} = 0$$

Initial conditions: $y(2) = 0, y'(2) = \frac{1}{2}$

$$0 = C_1 + C_2 \ln 2$$

$$y' = \frac{C_2}{x}$$

$$\frac{1}{2} = \frac{C_2}{2} \quad C_2 = 1, C_1 = -\ln 2$$

Particular solution: $y = -\ln 2 + \ln x = \ln \frac{x}{2}$

30. Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$$

$$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$$

$$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12\left(e^{2x/3}\right)\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4\left(e^{2x/3}\right)(C_1 + C_2x) = 0$$

Initial conditions: $y(0) = 4, y(3) = 0$

$$0 = e^2(C_1 + 3C_2)$$

$$4 = (1)(C_1 + 0) \quad C_1 = 4$$

$$0 = e^2(4 + 3C_2) \quad C_2 = -\frac{4}{3}$$

Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

32. $\frac{dy}{dx} = x^3 - 4x$

$$y = \int (x^3 - 4x) dx = \frac{x^4}{4} - 2x^2 + C$$

34. $\frac{dy}{dx} = \frac{e^x}{1 + e^x}$

$$y = \int \frac{e^x}{1 + e^x} dx = \ln(1 + e^x) + C$$

36. $\frac{dy}{dx} = x \cos x^2$

$$y = \int x \cos(x^2) dx = \frac{1}{2} \sin(x^2) + C$$

($u = x^2, du = 2x dx$)

38. $\frac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int (\sec^2 x - 1) dx = \tan x - x + C$$

40. $\frac{dy}{dx} = x\sqrt{5-x}$. Let $u = \sqrt{5-x}, u^2 = 5-x, dx = -2u du$

$$y = \int x\sqrt{5-x} dx = \int (5 - u^2)u(-2u) du$$

$$= \int (-10u^2 + 2u^4) du$$

$$= -\frac{10u^3}{3} + \frac{2u^5}{5} + C$$

$$= -\frac{10}{3}(5-x)^{3/2} + \frac{2}{5}(5-x)^{5/2} + C$$

42. $\frac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2} dx = 5(-2) \int e^{-x/2} \left(-\frac{1}{2}\right) dx$$

$$= -10e^{-x/2} + C$$

44. $\frac{dy}{dx} = \frac{x^2 + 2}{3y^2}$

$$\int 3y^2 dy = \int (x^2 + 2) dx$$

$$y^3 = \frac{x^3}{3} + 2x + C$$

46. $\frac{dr}{ds} = 0.05s$

$$\int dr = \int 0.05s ds$$

$$r = 0.025s^2 + C$$

48. $xy' = y$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

50. $y \frac{dy}{dx} = 6 \cos \pi x$

$$\int y dy = \int 6 \cos \pi x dx$$

$$\frac{y^2}{2} = \frac{6}{\pi} \sin \pi x + C_1$$

$$y^2 = \frac{12}{\pi} \sin \pi x + C$$

52. $\sqrt{x^2 - 9} \frac{dy}{dx} = 5x$

$$\int dy = \int \frac{5x}{\sqrt{x^2 - 9}} dx$$

$$y = 5(x^2 - 9)^{1/2} + C$$

54. $4y \frac{dy}{dx} = 3e^x$

$$\int 4y dy = \int 3e^x dx$$

$$2y^2 = 3e^x + C$$

56. $\sqrt{x} + \sqrt{y} y' = 0$

$$\int y^{1/2} dy = - \int x^{1/2} dx$$

$$\frac{2}{3} y^{3/2} = -\frac{2}{3} x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition: $y(1) = 4$,
 $(4)^{3/2} + (1)^{3/2} = 8 + 1 = 9 = C$

Particular solution: $y^{3/2} + x^{3/2} = 9$

58. $2xy' - \ln x^2 = 0$

$$2x \frac{dy}{dx} = 2 \ln x$$

$$\int dy = \int \frac{\ln x}{x} dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

$$y(1) = 2: 2 = C$$

$$y = \frac{1}{2}(\ln x)^2 + 2$$

60. $y\sqrt{1-x^2} \frac{dy}{dx} = x\sqrt{1-y^2}$

$$\int (1-y^2)^{-1/2} y dy = \int (1-x^2)^{-1/2} x dx$$

$$-(1-y^2)^{1/2} = -(1-x^2)^{1/2} + C$$

$$y(0) = 1: \quad 0 = -1 + C \quad C = 1$$

$$\sqrt{1-y^2} = \sqrt{1-x^2} - 1$$

62. $\frac{dr}{ds} = e^{r-2s}$

$$\int e^{-r} dr = \int e^{-2s} ds$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} + C$$

$$r(0) = 0: -1 = -\frac{1}{2} + C \quad C = -\frac{1}{2}$$

$$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$$

$$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$$

$$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1+e^{-2s}}{2}\right)$$

$$r = \ln\left(\frac{2}{1+e^{-2s}}\right)$$

64. $dT + k(T - 70) dt = 0$

$$\int \frac{dT}{T - 70} = -k \int dt$$

$$\ln(T - 70) = -kt + C_1$$

$$T - 70 = Ce^{-kt}$$

Initial condition: $T(0) = 140$;

$$140 - 70 = 70 = Ce^0 = C$$

Particular solution: $T - 70 = 70e^{-kt}$, $T = 70(1 + e^{-kt})$

68. $m = \frac{dy}{dx} = \frac{y-0}{x-0} = \frac{y}{x}$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

72. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{tx ty}{\sqrt{t^2 x^2 + t^2 y^2}}$$

$$= \frac{t^2 xy}{t\sqrt{x^2 + y^2}} = t \frac{xy}{\sqrt{x^2 + y^2}}$$

Homogeneous of degree 1

76. $f(x, y) = \tan \frac{y}{x}$

$$f(tx, ty) = \tan \frac{ty}{tx} = \tan \frac{y}{x}$$

Homogeneous of degree 0

78. $y' = \frac{(x^3 + y^3)}{xy^2}$

$$xy^2 dy = (x^3 + y^3) dx$$

$$y = vx, \quad dy = x dv + v dx$$

$$x(vx)^2(x dv + v dx) = (x^3 + (vx)^3) dx$$

$$x^4 v^2 dv + x^3 v^3 dx = x^3 dx + v^3 x^3 dx$$

$$xv^2 dv = dx$$

$$\int v^2 dv = \int \frac{1}{x} dx$$

$$\frac{v^3}{3} = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3 \ln|x| + C$$

$$y^3 = 3x^3 \ln|x| + Cx^3$$

66. $\frac{dy}{dx} = \frac{2y}{3x}$

$$\int \frac{3}{y} dy = \int \frac{2}{x} dx$$

$$\ln y^3 = \ln x^2 + \ln C$$

$$y^3 = Cx^2$$

Initial condition: $y(8) = 2$, $2^3 = C(8^2)$, $C = \frac{1}{8}$

Particular solution: $8y^3 = x^2$, $y = \frac{1}{2}x^{2/3}$

70. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

$$f(tx, ty) = t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

Not homogeneous

74. $f(x, y) = \tan(x + y)$

$$f(tx, ty) = \tan(tx + ty) = \tan[t(x + y)]$$

Not homogeneous

80. $y' = \frac{x^2 + y^2}{2xy}$, $y = vx$

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x^2 v}$$

$$2v dx + 2x dv = \frac{1 + v^2}{v} dx$$

$$\int \frac{2v}{v^2 - 1} dv = - \int \frac{dx}{x}$$

$$\ln(v^2 - 1) = -\ln x + \ln C = \ln \frac{C}{x}$$

$$v^2 - 1 = \frac{C}{x}$$

$$\frac{y^2}{x^2} - 1 = \frac{C}{x}$$

$$y^2 - x^2 = Cx$$

$$82. \quad y' = \frac{2x + 3y}{x}, y = vx$$

$$v + x \frac{dv}{dx} = \frac{2x + 3vx}{x} = 2 + 3v$$

$$x \frac{dv}{dx} = 2 + 2v \quad \int \frac{dv}{1+v} = 2 \int \frac{dx}{x}$$

$$\ln|1+v| = \ln x^2 + \ln C = \ln x^2 C$$

$$1+v = x^2 C$$

$$1 + \frac{y}{x} = x^2 C$$

$$\frac{y}{x} = x^2 C - 1$$

$$y = Cx^3 - x$$

$$84. \quad -y^2 dx + x(x+y) dy = 0, y = vx$$

$$-x^2 v^2 dx + (x^2 + x^2 v)(v dx + x dv) = 0$$

$$\int \frac{1+v}{v} dv = - \int \frac{dx}{x}$$

$$v + \ln v = -\ln x + \ln C_1 = \ln \frac{C_1}{x}$$

$$v = \ln \frac{C_1}{xv}$$

$$\frac{C_1}{vx} = e^v$$

$$\frac{C_1}{y} = e^{y/x}$$

$$y = Ce^{-y/x}$$

$$\text{Initial condition: } y(1) = 1, 1 = Ce^{-1} \quad C = e$$

$$\text{Particular solution: } y = e^{1-y/x}$$

$$86. \quad (2x^2 + y^2) dx + xy dy = 0$$

$$\text{Let } y = vx, dy = x dv + v dx.$$

$$(2x^2 + v^2 x^2) dx + x(vx)(x dv + v dx) = 0$$

$$(2x^2 + 2x^2 v^2) dx + x^3 v dv = 0$$

$$(2 + 2v^2) dx = -xv dv$$

$$\frac{-2}{x} dx = \frac{v}{1+v^2} dv$$

$$-2 \ln x = \frac{1}{2} \ln(1+v^2) + C_1$$

$$\ln x^{-2} = \ln(1+v^2)^{1/2} + \ln C$$

$$x^{-2} = C(1+v^2)^{1/2}$$

$$\frac{1}{x^2} = C \left(1 + \frac{y^2}{x^2}\right)^{1/2} = \frac{C}{x} (x^2 + y^2)^{1/2}$$

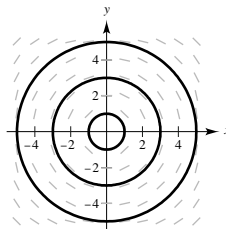
$$\frac{1}{x} = C(x^2 + y^2)^{1/2}$$

$$y(1) = 0: 1 = C(1+0) \quad C = 1$$

$$\frac{1}{x} = \sqrt{x^2 + y^2}$$

$$1 = x\sqrt{x^2 + y^2}$$

$$88. \quad \frac{dy}{dx} = -\frac{x}{y}$$



$$y dy = -x dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

90. $\frac{dy}{dx} = 0.25x(4 - y)$

$$\frac{dy}{4 - y} = 0.25x dx$$

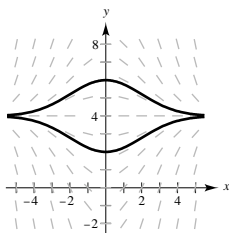
$$\int \frac{dy}{y - 4} = \int -0.25x dx$$

$$= = -\frac{1}{4} \int x dx$$

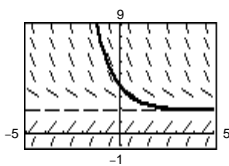
$$\ln |y - 4| = -\frac{1}{8}x^2 + C_1$$

$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

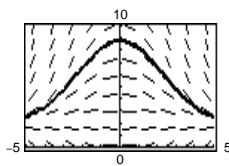
$$y = 4 + Ce^{-(1/8)x^2}$$



92. $\frac{dy}{dx} = 2 - y, y(0) = 4$



94. $\frac{dy}{dx} = 0.2x(2 - y), y(0) = 9$



96. $\frac{dy}{dt} = ky, y = Ce^{kt}$

Initial conditions: $y(0) = 20, y(1) = 16$

$$20 = Ce^0 = C$$

$$16 = 20e^k$$

$$k = \ln \frac{4}{5}$$

Particular solution: $y = 20e^{t \ln(4/5)}$

When 75% has been changed:

$$5 = 20e^{t \ln(4/5)}$$

$$\frac{1}{4} = e^{t \ln(4/5)}$$

$$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2 \text{ hr}$$

100. $\frac{dy}{dx} = ky^2$

The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

98. $\frac{dy}{dx} = k(x - 4)$

The direction field satisfies $(dy/dx) = 0$ along $x = 4$:
Matches (b).

102. From Exercise 101,

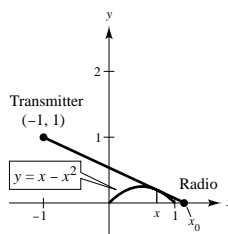
$$w = 1200 - Ce^{-kt}, k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \quad C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

104. Let the radio receiver be located at $(x_0, 0)$.
 The tangent line to $y = x - x^2$ joins $(-1, 1)$
 and $(x_0, 0)$.



- (a) If (x, y) is the point of tangency on the $y = x - x^2$, then

$$1 - 2x = \frac{y - 1}{x + 1} = \frac{x - x^2 - 1}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - 1$$

$$x^2 + 2x - 2 = 0$$

$$x = \left(\frac{-2 \pm \sqrt{4 + 8}}{2} \right) = -1 + \sqrt{3}$$

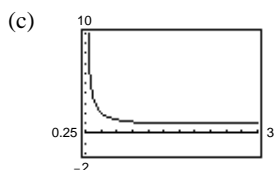
$$y = x - x^2 = 3\sqrt{3} - 5$$

$$\text{Then } \frac{1 - 0}{-1 - x_0} = \frac{1 - 3\sqrt{3} + 5}{-1 + 1 - \sqrt{3}} = \frac{6 - 3\sqrt{3}}{-\sqrt{3}}$$

$$\sqrt{3} = (1 + x_0)(6 - 3\sqrt{3})$$

$$= 6 - 3\sqrt{3} + x_0(6 - 3\sqrt{3})$$

$$x_0 = \frac{4\sqrt{3} - 6}{6 - 3\sqrt{3}} \approx 1.155$$



There is a vertical asymptote at $h = \frac{1}{4}$, which is the height of the mountain.

106. Given family (hyperbolas): $x^2 - 2y^2 = C$

$$2x - 4yy' = 0$$

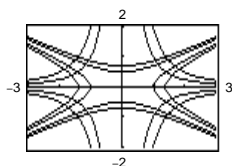
$$y' = \frac{x}{2y}$$

Orthogonal trajectory: $y' = -\frac{2y}{x}$

$$\int \frac{dy}{y} = -\int \frac{2}{x} dx$$

$$\ln y = -2 \ln x + \ln k$$

$$y = kx^{-2} = \frac{k}{x^2}$$



- (b) Now let the transmitter be located at $(-1, h)$.

$$1 - 2x = \frac{y - h}{x + 1} = \frac{x - x^2 - h}{x + 1}$$

$$x - 2x^2 + 1 - 2x = x - x^2 - h$$

$$x^2 + 2x - h - 1 = 0$$

$$x = \frac{(-2 \pm \sqrt{4 + 4(h + 1)})}{2}$$

$$= -1 + \sqrt{2 + h}$$

$$y = x - x^2$$

$$= 3\sqrt{2 + h} - h - 4$$

$$\text{Then, } \frac{h - 0}{-1 - x_0} = \frac{h - (3\sqrt{2 + h} - h - 4)}{-1 - (-1 + \sqrt{2 + h})}$$

$$= \frac{2h + 4 - 3\sqrt{2 + h}}{-\sqrt{2 + h}}$$

$$\frac{x_0 + 1}{h} = \frac{\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}}$$

$$x_0 = \frac{h\sqrt{2 + h}}{2h + 4 - 3\sqrt{2 + h}} - 1$$

108. Given family (parabolas): $y^2 = 2Cx$

$$2yy' = 2C$$

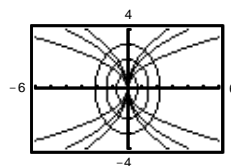
$$y' = \frac{C}{y} = \frac{y^2}{2x} \left(\frac{1}{y} \right) = \frac{y}{2x}$$

Orthogonal trajectory (ellipse): $y' = -\frac{2x}{y}$

$$\int y dy = -\int 2x dx$$

$$\frac{y^2}{2} = -x^2 + K_1$$

$$2x^2 + y^2 = K$$



110. Given family (exponential functions): $y = Ce^x$

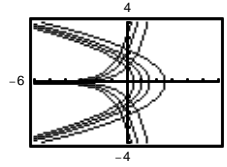
$$y' = Ce^x = y$$

Orthogonal trajectory (parabolas): $y' = -\frac{1}{y}$

$$\int y \, dy = -\int dx$$

$$\frac{y^2}{2} = -x + K_1$$

$$y^2 = -2x + K$$



112. The number of initial conditions matches the number of constants in the general solution.

114. Two families of curves are mutually orthogonal if each curve in the first family intersects each curve in the second family at right angles.

116. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

118. True

$$x^2 + y^2 = 2Cy$$

$$x^2 + y^2 = 2Kx$$

$$\frac{dy}{dx} = \frac{x}{C - y}$$

$$\frac{dy}{dx} = \frac{K - x}{y}$$

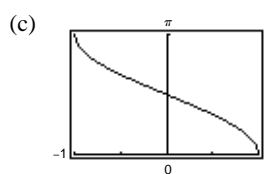
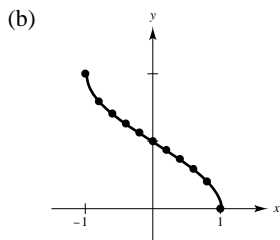
$$\frac{x}{C - y} \cdot \frac{K - x}{y} = \frac{Kx - x^2}{Cy - y^2} = \frac{2Kx - 2x^2}{2Cy - 2y^2} = \frac{x^2 + y^2 - 2x^2}{x^2 + y^2 - 2y^2} = \frac{y^2 - x^2}{x^2 - y^2} = -1$$

Section 5.8 Inverse Trigonometric Functions: Differentiation

2. $y = \arccos x$

(a)

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	3.142	2.499	2.214	1.982	1.772	1.571	1.369	1.159	0.927	0.634	0



(d) Intercepts: $(0, \frac{\pi}{2})$ and $(1, 0)$

No symmetry

4. $(_, \frac{\pi}{4}) = (1, \frac{\pi}{4})$

$$(_, -\frac{\pi}{6}) = (-\frac{\sqrt{3}}{3}, -\frac{\pi}{6})$$

$$(-\sqrt{3}, _) = (-\sqrt{3}, -\frac{\pi}{3})$$

6. $\arcsin 0 = 0$

8. $\arccos 0 = \frac{\pi}{2}$

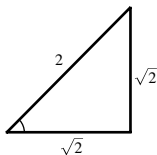
10. $\arccot(-\sqrt{3}) = \frac{5\pi}{6}$

12. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

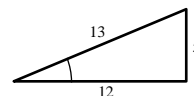
14. $\arcsin(-0.39) \approx -0.40$

16. $\arctan(-3) \approx -1.25$

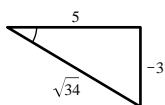
18. (a) $\tan\left(\arccos\frac{\sqrt{2}}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$



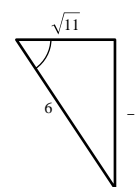
(b) $\cos\left(\arcsin\frac{5}{13}\right) = \frac{12}{13}$



20. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right] = \frac{\sqrt{34}}{5}$



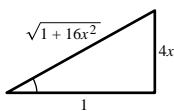
(b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right] = -\frac{5\sqrt{11}}{11}$



22. $y = \sec(\arctan 4x)$

$\theta = \arctan 4x$

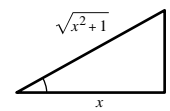
$y = \sec \theta = \sqrt{1 + 16x^2}$



24. $y = \cos(\operatorname{arccot} x)$

$\theta = \operatorname{arccot} x$

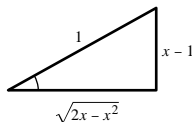
$y = \cos \theta = \frac{x}{\sqrt{x^2 + 1}}$



26. $y = \sec[\arcsin(x - 1)]$

$\theta = \arcsin(x - 1)$

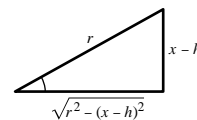
$y = \sec \theta = \frac{1}{\sqrt{2x - x^2}}$



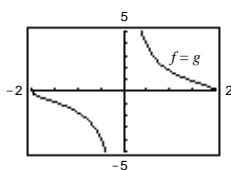
28. $y = \cos\left(\arcsin\frac{x-h}{r}\right)$

$\theta = \arcsin\frac{x-h}{r}$

$y = \cos \theta = \frac{\sqrt{r^2 - (x-h)^2}}{r}$



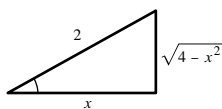
30.


 Asymptote: $x = 0$

$\arccos\frac{x}{2} = \theta$

$\cos \theta = \frac{x}{2}$

$\tan \theta = \frac{\sqrt{4-x^2}}{x}$



32. $\arctan(2x - 5) = -1$

$2x - 5 = \tan(-1)$

$x = \frac{1}{2}(\tan(-1) + 5) \approx 1.721$

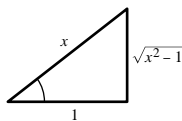
34. $\arccos x = \operatorname{arcsec} x$

$$x = \cos(\operatorname{arcsec} x)$$

$$x = \frac{1}{x}$$

$$x^2 = 1$$

$$x = \pm 1$$



36. (a) $\arcsin(-x) = -\arcsin x, |x| \leq 1$.

Let $y = \arcsin(-x)$. Then,

$$-x = \sin y \quad x = -\sin y \quad x = \sin(-y).$$

Thus, $-y = \arcsin x$ $y = -\arcsin x$. Therefore, $\arcsin(-x) = -\arcsin x$.

(b) $\operatorname{arccos}(-x) = \pi - \operatorname{arccos} x, |x| \leq 1$.

Let $y = \operatorname{arccos}(-x)$. Then,

$$-x = \cos y \quad x = -\cos y \quad x = \cos(\pi - y).$$

Thus, $\pi - y = \operatorname{arccos} x$ $y = \pi - \operatorname{arccos} x$. Therefore, $\operatorname{arccos}(-x) = \pi - \operatorname{arccos} x$.

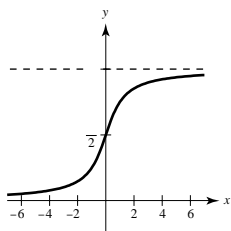
38. $f(x) = \arctan x + \frac{\pi}{2}$

$$x = \tan\left(y - \frac{\pi}{2}\right)$$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

$f(x)$ is the graph of $\arctan x$ shifted $\pi/2$ units upward.



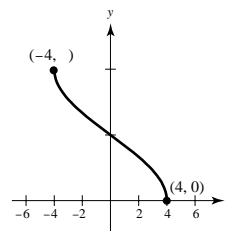
40. $f(x) = \operatorname{arccos}\left(\frac{x}{4}\right)$

$$\frac{x}{4} = \cos y$$

$$x = 4 \cos y$$

Domain: $[-4, 4]$

Range: $[0, \pi]$



42. $f(t) = \arcsin t^2$

$$f'(t) = \frac{2t}{\sqrt{1-t^4}}$$

46. $f(x) = \arctan \sqrt{x}$

$$f'(x) = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}$$

50. $f(x) = \arcsin x + \operatorname{arccos} x = \frac{\pi}{2}$

$$f'(x) = 0$$

44. $f(x) = \operatorname{arcsec} 2x$

$$f'(x) = \frac{2}{|2x|\sqrt{4x^2-1}} = \frac{1}{|x|\sqrt{4x^2-1}}$$

48. $h(x) = x^2 \arctan x$

$$h'(x) = 2x \arctan x + \frac{x^2}{1+x^2}$$

52. $y = \ln(t^2 + 4) - \frac{1}{2} \arctan \frac{t}{2}$

$$\begin{aligned} y' &= \frac{2t}{t^2+4} - \frac{1}{2} \cdot \frac{1}{1+\left(\frac{t}{2}\right)^2} \left(\frac{1}{2}\right) \\ &= \frac{2t}{t^2+4} - \frac{1}{t^2+4} = \frac{2t-1}{t^2+4} \end{aligned}$$

54. $y = \frac{1}{2} \left[x\sqrt{4-x^2} + 4 \arcsin\left(\frac{x}{2}\right) \right]$

$$\begin{aligned} y' &= \frac{1}{2} \left[x \frac{1}{2} (4-x^2)^{-1/2} (-2x) + \sqrt{4-x^2} + 2 \frac{1}{\sqrt{1-(x/2)^2}} \right] \\ &= \frac{1}{2} \left[\frac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} + \frac{4}{\sqrt{4-x^2}} \right] \\ &= \sqrt{4-x^2} \end{aligned}$$

56. $y = x \arctan 2x - \frac{1}{4} \ln(1+4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4} \left(\frac{8x}{1+4x^2} \right) = \arctan(2x)$$

58. $y = 25 \arcsin \frac{x}{5} - x\sqrt{25 - x^2}$

$$\begin{aligned} y' &= 5 \frac{1}{\sqrt{1 - (x/5)^2}} - \sqrt{25 - x^2} - x \frac{1}{2} (25 - x^2)^{-1/2} (-2x) \\ &= \frac{25}{\sqrt{25 - x^2}} - \frac{(25 - x^2)}{\sqrt{25 - x^2}} + \frac{x^2}{\sqrt{25 - x^2}} \\ &= \frac{2x^2}{\sqrt{25 - x^2}} \end{aligned}$$

60. $y = \arctan \frac{x}{2} - \frac{1}{2(x^2 + 4)}$

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{1 + (x/2)^2} + \frac{1}{2} (x^2 + 4)^{-2} (2x) \\ &= \frac{2}{x^2 + 4} + \frac{x}{(x^2 + 4)^2} \\ &= \frac{2x^2 + 8 + x}{(x^2 + 4)^2} \end{aligned}$$

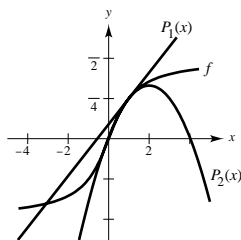
62. $f(x) = \arctan x, a = 1$

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2} f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$



64. $f(x) = \arcsin x - 2x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - 2 = 0 \text{ when } \sqrt{1 - x^2} = \frac{1}{2} \text{ or}$$

$$x = \pm \frac{\sqrt{3}}{2}$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$f''\left(\frac{\sqrt{3}}{2}\right) > 0$$

Relative minimum: $\left(\frac{\sqrt{3}}{2}, -0.68\right)$

$$f''\left(-\frac{\sqrt{3}}{2}\right) < 0$$

Relative maximum: $\left(-\frac{\sqrt{3}}{2}, 0.68\right)$

66. $f(x) = \arcsin x - 2 \arctan x$

$$f'(x) = \frac{1}{\sqrt{1 - x^2}} - \frac{2}{1 + x^2} = 0$$

$$1 + x^2 = 2\sqrt{1 - x^2}$$

$$1 + 2x^2 + x^4 = 4(1 - x^2)$$

$$x^4 + 6x^2 - 3 = 0$$

$$x = \pm 0.681$$

By the First Derivative Test, $(-0.681, 0.447)$ is a relative maximum and $(0.681, -0.447)$ is a relative minimum.

68. $\arctan 0 = 0$. π is not in the range of $y = \arctan x$.

70. The derivatives are algebraic. See Theorem 5.18.

72. (a) $\cot \theta = \frac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\frac{d\theta}{dt} = \frac{-3}{x^2 + 9} \frac{dx}{dt}$

If $x = 10$, $\frac{d\theta}{dt} \approx 11.001$ rad/hr.

If $x = 3$, $\frac{d\theta}{dt} \approx 66.667$ rad/hr.

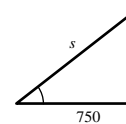
A lower altitude results in a greater rate of change of θ .

74. $\cos \theta = \frac{750}{s}$

$$\theta = \arccos\left(\frac{750}{s}\right)$$

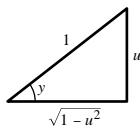
$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \frac{-1}{\sqrt{1 - (750/s)^2}} \left(\frac{-750}{s^2}\right) \frac{ds}{dt}$$

$$= \frac{750}{s\sqrt{s^2 - 750^2}} \frac{ds}{dt}$$

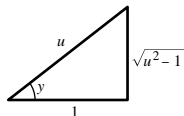


76. (a) Let $y = \arcsin u$. Then

$$\begin{aligned}\sin y &= u \\ \cos y \cdot y' &= u' \\ \frac{dy}{dx} &= \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}.\end{aligned}$$


 (c) Let $y = \operatorname{arcsec} u$. Then

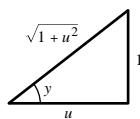
$$\begin{aligned}\sec y &= u \\ \sec y \tan y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}.\end{aligned}$$



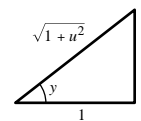
Note: The absolute value sign in the formula for the derivative of $\operatorname{arcsec} u$ is necessary because the inverse secant function has a positive slope at every value in its domain.

 (e) Let $y = \operatorname{arccot} u$. Then

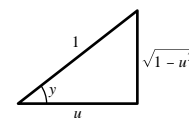
$$\begin{aligned}\cot y &= u \\ -\csc^2 y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{-\csc^2 y} = -\frac{u'}{1+u^2}.\end{aligned}$$


 (b) Let $y = \arctan u$. Then

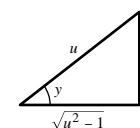
$$\begin{aligned}\tan y &= u \\ \sec^2 y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}.\end{aligned}$$


 (d) Let $y = \arccos u$. Then

$$\begin{aligned}\cos y &= u \\ -\sin y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}.\end{aligned}$$


 (f) Let $y = \operatorname{arccsc} u$. Then

$$\begin{aligned}\csc y &= u \\ -\csc y \cot y \frac{dy}{dx} &= u' \\ \frac{dy}{dx} &= \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2-1}}.\end{aligned}$$

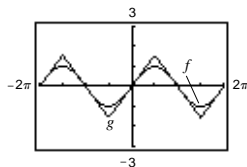


Note: The absolute value sign in the formula for the derivative of $\operatorname{arccsc} u$ is necessary because the inverse cosecant function has a negative slope at every value in its domain.

 78. $f(x) = \sin x$
 $g(x) = \arcsin(\sin x)$

 (a) The range of $y = \arcsin x$ is $-\pi/2 \leq y \leq \pi/2$.

 (b) Maximum: $\pi/2$

 Minimum: $-\pi/2$


80. False

 The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

82. False

 $\arcsin^2 0 + \arccos^2 0 = 0 + \left(\frac{\pi}{2}\right)^2 \neq 1$

Section 5.9 Inverse Trigonometric Functions: Integration

2. $\int \frac{3}{\sqrt{1-4x^2}} dx = \frac{3}{2} \int \frac{2}{\sqrt{1-4x^2}} dx = \frac{3}{2} \arcsin(2x) + C$

4. $\int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$

6. $\int \frac{4}{1+9x^2} dx = \frac{4}{3} \int \frac{3}{1+9x^2} dx = \frac{4}{3} \arctan(3x) + C$

8. $\int_{\sqrt{3}}^3 \frac{1}{\sqrt{9+x^2}} dx = \left[\frac{1}{3} \arctan \frac{x}{3} \right]_{\sqrt{3}}^3 = \frac{\pi}{36}$

10. $\int \frac{1}{4+(x-1)^2} dx = \frac{1}{2} \arctan\left(\frac{x-1}{2}\right) + C$

12. $\int \frac{x^4-1}{x^2+1} dx = \int (x^2-1) dx = \frac{1}{3}x^3 - x + C$

14. Let $u = t^2$, $du = 2t dt$.

$$\int \frac{t}{t^4 + 16} dt = \frac{1}{2} \int \frac{1}{(4)^2 + (t^2)^2} (2t) dt = \frac{1}{8} \arctan \frac{t^2}{4} + C$$

18. Let $u = \arccos x$, $du = -\frac{1}{\sqrt{1-x^2}} dx$.

$$\begin{aligned} \int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}} dx &= - \int_0^{1/\sqrt{2}} \frac{-\arccos x}{\sqrt{1-x^2}} dx \\ &= \left[-\frac{1}{2} \arccos^2 x \right]_0^{1/\sqrt{2}} = \frac{3\pi^2}{32} \approx 0.925 \end{aligned}$$

22.
$$\int_1^2 \frac{1}{3 + (x-2)^2} dx = \int_1^2 \frac{1}{(\sqrt{3})^2 + (x-2)^2} dx = \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) \right]_1^2 = \frac{\sqrt{3}\pi}{18}$$

24.
$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx = \arctan(\sin x) \Big|_0^{\pi/2} = \frac{\pi}{4}$$

16. Let $u = x^2$, $du = 2x dx$.

$$\begin{aligned} \int \frac{1}{x\sqrt{x^4-4}} dx &= \frac{1}{2} \int \frac{1}{x^2\sqrt{(x^2)^2-2^2}} (2x) dx \\ &= \frac{1}{4} \operatorname{arcsec} \frac{x^2}{2} + C \end{aligned}$$

20. Let $u = 1 + x^2$, $du = 2x dx$.

$$\begin{aligned} \int_{-\sqrt{3}}^0 \frac{x}{1+x^2} dx &= \frac{1}{2} \int_{-\sqrt{3}}^0 \frac{1}{1+x^2} (2x) dx \\ &= \left[\frac{1}{2} \ln(1+x^2) \right]_{-\sqrt{3}}^0 = -\ln 2 \end{aligned}$$

26.
$$\begin{aligned} \int \frac{3}{2\sqrt{x}(1+x)} dx, u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx, dx = 2u du \\ \frac{3}{2} \int \frac{2u du}{u(1+u^2)} = 3 \int \frac{du}{1+u^2} = 3 \arctan u + C \\ = 3 \arctan \sqrt{x} + C \end{aligned}$$

28.
$$\int \frac{4x+3}{\sqrt{1-x^2}} dx = (-2) \int \frac{-2x}{\sqrt{1-x^2}} dx + 3 \int \frac{1}{\sqrt{1-x^2}} dx = -4\sqrt{1-x^2} + 3 \arcsin x + C$$

30.
$$\begin{aligned} \int \frac{x-2}{(x+1)^2+4} dx &= \frac{1}{2} \int \frac{2x+2}{(x+1)^2+4} dx - \int \frac{3}{(x+1)^2+4} dx \\ &= \frac{1}{2} \ln(x^2+2x+5) - \frac{3}{2} \arctan\left(\frac{x+1}{2}\right) + C \end{aligned}$$

32.
$$\int_{-2}^2 \frac{dx}{x^2+4x+13} = \int_{-2}^2 \frac{dx}{(x+2)^2+9} = \left[\frac{1}{3} \arctan\left(\frac{x+2}{3}\right) \right]_{-2}^2 = \frac{1}{3} \arctan\left(\frac{4}{3}\right)$$

34.
$$\int \frac{2x-5}{x^2+2x+2} dx = \int \frac{2x+2}{x^2+2x+2} dx - 7 \int \frac{1}{1+(x+1)^2} dx = \ln|x^2+2x+2| - 7 \arctan(x+1) + C$$

36.
$$\begin{aligned} \int \frac{2}{\sqrt{-x^2+4x}} dx &= \int \frac{2}{\sqrt{4-(x^2-4x+4)}} dx \\ &= \int \frac{2}{\sqrt{4-(x-2)^2}} dx \\ &= 2 \arcsin\left(\frac{x-2}{2}\right) + C \end{aligned}$$

38. Let $u = x^2 - 2x$, $du = (2x - 2) dx$.

$$\begin{aligned} \int \frac{x-1}{\sqrt{x^2-2x}} dx &= \frac{1}{2} \int (x^2-2x)^{-1/2} (2x-2) dx \\ &= \sqrt{x^2-2x} + C \end{aligned}$$

40.
$$\int \frac{1}{(x-1)\sqrt{x^2-2x}} dx = \int \frac{1}{(x-1)\sqrt{(x-1)^2-1}} dx = \operatorname{arcsec}|x-1| + C$$

42. Let $u = x^2 - 4$, $du = 2x dx$.

$$\int \frac{x}{\sqrt{9 + 8x^2 - x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{25 - (x^2 - 4)^2}} dx = \frac{1}{2} \arcsin\left(\frac{x^2 - 4}{5}\right) + C$$

44. Let $u = \sqrt{x - 2}$, $u^2 + 2 = x$, $2u du = dx$

$$\begin{aligned} \int \frac{\sqrt{x-2}}{x+1} dx &= \int \frac{2u^2}{u^2+3} du = \int \frac{2u^2+6-6}{u^2+3} du = 2 \int du - 6 \int \frac{1}{u^2+3} du \\ &= 2u - \frac{6}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C = 2\sqrt{x-2} - 2\sqrt{3} \arctan \sqrt{\frac{x-2}{3}} + C \end{aligned}$$

46. The term is $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$: $x^2 + 3x = x^2 + 3x + \frac{9}{4} - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$

48. (a) $\int e^{x^2} dx$ cannot be evaluated using the basic integration rules.

(b) $\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$, $u = x^2$

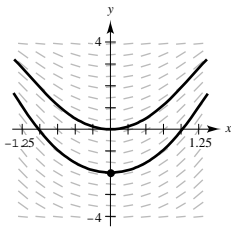
(c) $\int \frac{1}{x^2} e^{1/x} dx = -e^{1/x} + C$, $u = \frac{1}{x}$

50. (a) $\int \frac{1}{1+x^4} dx$ cannot be evaluated using the basic integration rules.

(b) $\int \frac{x}{1+x^4} dx = \frac{1}{2} \int \frac{2x}{1+(x^2)^2} dx = \frac{1}{2} \arctan(x^2) + C$, $u = x^2$

(c) $\int \frac{x^3}{1+x^4} dx = \frac{1}{4} \int \frac{4x^3}{1+x^4} dx = \frac{1}{4} \ln(1+x^4) + C$, $u = 1+x^4$

52. (a)



(b) $\frac{dy}{dx} = x\sqrt{16-y^2}$, $(0, -2)$

$$\frac{dy}{\sqrt{16-y^2}} = x dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} + C$$

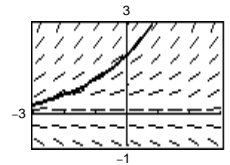
$(0, -2)$: $\arcsin\left(-\frac{2}{4}\right) = C$ $C = -\frac{\pi}{6}$

$$\arcsin\left(\frac{y}{4}\right) = \frac{x^2}{2} - \frac{\pi}{6}$$

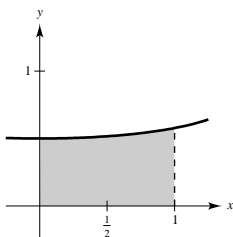
$$\frac{y}{4} = \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

$$y = 4 \sin\left(\frac{x^2}{2} - \frac{\pi}{6}\right)$$

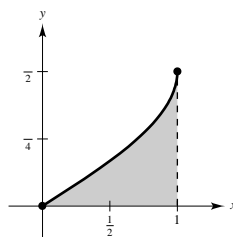
54. $\frac{dy}{dx} = \frac{2y}{\sqrt{16-x^2}}$, $y(0) = 2$



$$56. A = \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}$$



$$58. \int_0^1 \arcsin x dx \approx 0.571$$



$$60. F(x) = \frac{1}{2} \int_x^{x+2} \frac{2}{t^2+1} dt$$

(a) $F(x)$ represents the average value of $f(x)$ over the interval $[x, x+2]$. Maximum at $x = -1$, since the graph is greatest on $[-1, 1]$.

$$(b) F(x) = \left[\arctan t \right]_x^{x+2} = \arctan(x+2) - \arctan x$$

$$F'(x) = \frac{1}{1+(x+2)^2} - \frac{1}{1+x^2} = \frac{(1+x^2) - (x^2+4x+5)}{(x^2+1)(x^2+4x+5)} = \frac{-4(x+1)}{(x^2+1)(x^2+4x+5)} = 0 \text{ when } x = -1.$$

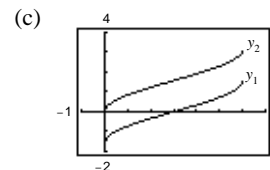
$$62. \int \frac{1}{\sqrt{6x-x^2}} dx$$

$$(a) 6x - x^2 = 9 - (x^2 - 6x + 9) = 9 - (x-3)^2$$

$$\int \frac{1}{\sqrt{6x-x^2}} dx = \int \frac{dx}{\sqrt{9-(x-3)^2}} = \arcsin\left(\frac{x-3}{3}\right) + C$$

$$(b) u = \sqrt{x}, u^2 = x, 2u du = dx$$

$$\int \frac{1}{\sqrt{6u^2-u^4}} (2u du) = \int \frac{2}{\sqrt{6-u^2}} du = 2 \arcsin\left(\frac{u}{\sqrt{6}}\right) + C = 2 \arcsin\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$



The antiderivatives differ by a constant, $\pi/2$.

Domain: $[0, 6]$

$$64. \text{ Let } f(x) = \arctan x - \frac{x}{1+x^2}$$

$$f'(x) = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \text{ for } x > 0.$$

Since $f(0) = 0$ and f is increasing for $x > 0$, $\arctan x - \frac{x}{1+x^2} > 0$ for $x > 0$. Thus,

$$\arctan x > \frac{x}{1+x^2}.$$

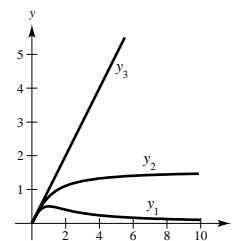
$$\text{ Let } g(x) = x - \arctan x$$

$$g'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0 \text{ for } x > 0.$$

Since $g(0) = 0$ and g is increasing for $x > 0$, $x - \arctan x > 0$ for $x > 0$. Thus, $x > \arctan x$.

Therefore,

$$\frac{x}{1+x^2} < \arctan x < x.$$



Section 5.10 Hyperbolic Functions

2. (a) $\cosh(0) = \frac{e^0 + e^0}{2} = 1$

(b) $\operatorname{sech}(1) = \frac{2}{e + e^{-1}} \approx 0.648$

4. (a) $\sinh^{-1}(0) = 0$

(b) $\tanh^{-1}(0) = 0$

6. (a) $\operatorname{csch}^{-1}(2) = \ln\left(\frac{1 + \sqrt{5}}{2}\right) \approx 0.481$

(b) $\operatorname{coth}^{-1}(3) = \frac{1}{2} \ln\left(\frac{4}{2}\right) \approx 0.347$

8. $\frac{1 + \cosh 2x}{2} = \frac{1 + (e^{2x} + e^{-2x})/2}{2} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2 x$

10. $2 \sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

12. $2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right) = 2 \left[\frac{e^{(x+y)/2} + e^{-(x+y)/2}}{2} \right] \left[\frac{e^{(x-y)/2} + e^{-(x-y)/2}}{2} \right]$
 $= 2 \left[\frac{e^x + e^y + e^{-y} + e^{-x}}{4} \right] = \frac{e^x + e^{-x}}{2} + \frac{e^y + e^{-y}}{2}$
 $= \cosh x + \cosh y$

14. $\tanh x = \frac{1}{2}$

$\left(\frac{1}{2}\right)^2 + \operatorname{sech}^2 x = 1 \quad \operatorname{sech}^2 x = \frac{3}{4} \quad \operatorname{sech} x = \frac{\sqrt{3}}{2}$

$\cosh x = \frac{1}{\sqrt{3/2}} = \frac{2\sqrt{3}}{3}$

$\operatorname{coth} x = \frac{1}{1/2} = 2$

$\sinh x = \tanh x \cosh x = \left(\frac{1}{2}\right)\left(\frac{2\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$

$\operatorname{csch} x = \frac{1}{\sqrt{3}/3} = \sqrt{3}$

Putting these in order:

$\sinh x = \frac{\sqrt{3}}{3} \quad \operatorname{csch} x = \sqrt{3}$

$\cosh x = \frac{2\sqrt{3}}{3} \quad \operatorname{sech} x = \frac{\sqrt{3}}{2}$

$\tanh x = \frac{1}{2} \quad \operatorname{coth} x = 2$

16. $y = \operatorname{coth}(3x)$

$y' = -3 \operatorname{csch}^2(3x)$

18. $g(x) = \ln(\cosh x)$

$g'(x) = \frac{1}{\cosh x}(\sinh x) = \tanh x$

20. $y = x \cosh x - \sinh x$

$y' = x \sinh x + \cosh x - \cosh x = x \sinh x$

22. $h(t) = t - \operatorname{coth} t$

$h'(t) = 1 + \operatorname{csch}^2 t = \operatorname{coth}^2 t$

24. $g(x) = \operatorname{sech}^2 3x$

$$g'(x) = -2 \operatorname{sech}(3x) \operatorname{sech}(3x) \tanh(3x)(3)$$

$$= -6 \operatorname{sech}^2 3x \tanh 3x$$

26. $f(x) = e^{\sinh x}$

$$f'(x) = (\cosh x)(e^{\sinh x})$$

28. $y = \operatorname{sech}(x + 1)$

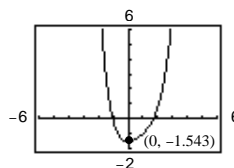
$$y' = -\operatorname{sech}(x + 1) \tanh(x + 1)$$

30. $f(x) = x \sinh(x - 1) - \cosh(x - 1)$

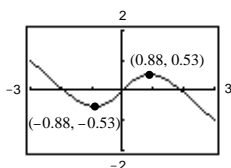
$$f'(x) = x \cosh(x - 1) + \sinh(x - 1) - \sinh(x - 1) = x \cosh(x - 1)$$

$$f'(x) = 0 \text{ for } x = 0. \text{ By the First Derivative}$$

Test, $(0, -\cosh(-1)) \approx (0, -1.543)$ is a relative minimum.



32. $h(x) = 2 \tanh x - x$



Relative maximum: $(0.88, 0.53)$

Relative minimum: $(-0.88, -0.53)$

34. $y = a \cosh x$

$$y' = a \sinh x$$

$$y'' = a \cosh x$$

Therefore, $y'' - y = 0$.

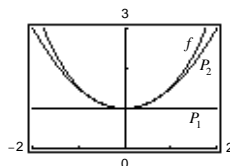
36. $f(x) = \cosh x \quad f(1) = \cosh(0) \approx 1$

$$f'(x) = \sinh x \quad f'(1) = \sinh(0) \approx 0$$

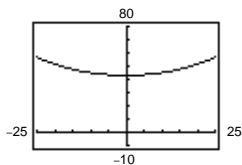
$$f''(x) = \cosh x \quad f''(1) = \cosh(0) \approx 1$$

$$P_1(x) = f(0) + f'(0)(x - 0) = 1$$

$$P_2(x) = 1 + \frac{1}{2}x^2$$



38. (a) $y = 18 + 25 \cosh \frac{x}{25}, -25 \leq x \leq 25$



(b) At $x = \pm 25, y = 18 + 25 \cosh(1) \approx 56.577$.

At $x = 0, y = 18 + 25 = 43$.

(c) $y' = \sinh \frac{x}{25}$. At $x = 25, y' = \sinh(1) \approx 1.175$

40. Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx$.

$$\int \frac{\cosh \sqrt{x}}{\sqrt{x}} dx = 2 \int \cosh \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right) dx = 2 \sinh \sqrt{x} + C$$

42. Let $u = \cosh x, du = \sinh x dx$.

$$\int \frac{\sinh x}{1 + \sinh^2 x} dx = \int \frac{\sinh x}{\cosh^2 x} dx = \frac{-1}{\cosh x} + C$$

$$= -\operatorname{sech} x + C$$

44. Let $u = 2x - 1, du = 2 dx$.

$$\int \operatorname{sech}^2(2x - 1) dx = \frac{1}{2} \int \operatorname{sech}^2(2x - 1)(2) dx$$

$$= \frac{1}{2} \tanh(2x - 1) + C$$

46. Let $u = \operatorname{sech} x, du = -\operatorname{sech} x \tanh x dx$.

$$\int \operatorname{sech}^3 x \tanh x dx = - \int \operatorname{sech}^2 x (-\operatorname{sech} x \tanh x) dx$$

$$= -\frac{1}{3} \operatorname{sech}^3 x + C$$

$$\begin{aligned}
 48. \int \cosh^2 x \, dx &= \int \frac{1 + \cosh 2x}{2} \, dx \\
 &= \frac{1}{2} \left[x + \frac{\sinh 2x}{2} \right] + C \\
 &= \frac{1}{2}x + \frac{1}{4} \sinh 2x + C
 \end{aligned}$$

$$50. \int_0^4 \frac{1}{\sqrt{25-x^2}} \, dx = \left[\arcsin \frac{x}{5} \right]_0^4 = \arcsin \frac{4}{5}$$

$$52. \int \frac{2}{x\sqrt{1+4x^2}} \, dx = 2 \int \frac{1}{(2x)\sqrt{1+(2x)^2}} (2) \, dx = -2 \ln \left(\frac{1 + \sqrt{1+4x^2}}{|2x|} \right) + C$$

$$54. \text{ Let } u = \sinh x, \, du = \cosh x \, dx.$$

$$\begin{aligned}
 \int \frac{\cosh x}{\sqrt{9 - \sinh^2 x}} \, dx &= \arcsin \left(\frac{\sinh x}{3} \right) + C \\
 &= \arcsin \left(\frac{e^x - e^{-x}}{6} \right) + C
 \end{aligned}$$

$$56. \, y = \tanh^{-1} \left(\frac{x}{2} \right)$$

$$y' = \frac{1}{1 - (x/2)^2} \left(\frac{1}{2} \right) = \frac{2}{4 - x^2}$$

$$58. \, y = \operatorname{sech}^{-1}(\cos 2x), \, 0 < x < \frac{\pi}{4}$$

$$y' = \frac{-1}{\cos 2x \sqrt{1 - \cos^2 2x}} (-2 \sin 2x) = \frac{2 \sin 2x}{\cos 2x |\sin 2x|} = \frac{2}{\cos 2x} = 2 \sec 2x,$$

since $\sin 2x > 0$ for $0 < x < \pi/4$.

$$60. \, y = (\operatorname{csch}^{-1} x)^2$$

$$y' = 2 \operatorname{csch}^{-1} x \left(\frac{-1}{|x| \sqrt{1+x^2}} \right) = \frac{-2 \operatorname{csch}^{-1} x}{|x| \sqrt{1+x^2}}$$

$$62. \, y = x \tanh^{-1} x + \ln \sqrt{1-x^2} = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2)$$

$$y' = x \left(\frac{1}{1-x^2} \right) + \tanh^{-1} x + \frac{-x}{1-x^2} = \tanh^{-1} x$$

64. See page 401, Theorem 5.22.

66. Equation of tangent line through $P = (x_0, y_0)$:

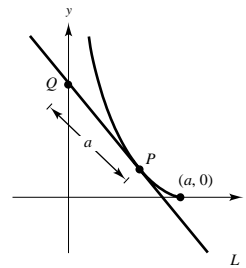
$$y - a \operatorname{sech}^{-1} \frac{x_0}{a} + \sqrt{a^2 - x_0^2} = -\frac{\sqrt{a^2 - x_0^2}}{x_0} (x - x_0)$$

When $x = 0$,

$$y = a \operatorname{sech}^{-1} \frac{x_0}{a} - \sqrt{a^2 - x_0^2} + \sqrt{a^2 - x_0^2} = a \operatorname{sech}^{-1} \frac{x_0}{a}.$$

Hence, Q is the point $[0, a \operatorname{sech}^{-1}(x_0/a)]$.

Distance from P to Q : $d = \sqrt{x_0^2 + (-\sqrt{a^2 - x_0^2})^2} = a$



$$\begin{aligned}
 68. \int \frac{x}{9-x^4} \, dx &= -\frac{1}{2} \int \frac{-2x}{9-(x^2)^2} \, dx = -\frac{1}{2} \left(\frac{1}{6} \right) \ln \left| \frac{3-x^2}{3+x^2} \right| + C \\
 &= -\frac{1}{12} \ln \left| \frac{3-x^2}{3+x^2} \right| + C
 \end{aligned}$$

70. Let $u = x^{3/2}$, $du = \frac{3}{2}\sqrt{x} dx$.

$$\int \frac{\sqrt{x}}{\sqrt{1+x^3}} dx = \frac{2}{3} \int \frac{1}{\sqrt{1+(x^{3/2})^2}} \left(\frac{3}{2}\sqrt{x}\right) dx = \frac{2}{3} \sinh^{-1}(x^{3/2}) + C = \frac{2}{3} \ln(x^{3/2} + \sqrt{1+x^3}) + C$$

72.
$$\int \frac{dx}{(x+2)\sqrt{x^2+4x+8}} = \int \frac{dx}{(x+2)\sqrt{(x+2)^2+4}}$$

$$= -\frac{1}{2} \ln\left(\frac{2 + \sqrt{(x+2)^2+4}}{|x+2|}\right) + C$$

74.
$$\int \frac{1}{(x+1)\sqrt{2x^2+4x+8}} dx = \int \frac{1}{(x+1)\sqrt{2(x+1)^2+6}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{(x+1)\sqrt{(x+1)^2+(\sqrt{3})^2}} dx = -\frac{1}{\sqrt{6}} \ln\left(\frac{\sqrt{3} + \sqrt{(x+1)^2+3}}{x+1}\right) + C$$

76. Let $u = 2(x-1)$, $du = 2 dx$.

$$y = \int \frac{1}{(x-1)\sqrt{-4x^2+8x-1}} dx = \int \frac{2}{2(x-1)\sqrt{(\sqrt{3})^2 - [2(x-1)]^2}} dx = -\frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{3} + \sqrt{-4x^2+8x-1}}{2(x-1)}\right| + C$$

78.
$$y = \int \frac{1-2x}{4x-x^2} dx = \int \frac{4-2x}{4x-x^2} dx + 3 \int \frac{1}{(x-2)^2-4} dx$$

$$= \ln|4x-x^2| + \frac{3}{4} \ln\left|\frac{(x-2)-2}{(x-2)+2}\right| + C = \ln|4x-x^2| + \frac{3}{4} \ln\left|\frac{x-4}{x}\right| + C$$

80.
$$A = \int_0^2 \tanh 2x dx$$

$$= \int_0^2 \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$$

$$= \frac{1}{2} \int_0^2 \frac{1}{e^{2x} + e^{-2x}} (2)(e^{2x} - e^{-2x}) dx$$

$$= \left[\frac{1}{2} \ln(e^{2x} + e^{-2x})\right]_0^2$$

$$= \frac{1}{2} \ln(e^4 + e^{-4}) - \frac{1}{2} \ln 2$$

$$= \ln \sqrt{\frac{e^4 + e^{-4}}{2}} \approx 1.654$$

82.
$$A = \int_3^5 \frac{6}{\sqrt{x^2-4}} dx$$

$$= \left[6 \ln(x + \sqrt{x^2-4})\right]_3^5$$

$$= 6 \ln(5 + \sqrt{21}) - 6 \ln(3 + \sqrt{5})$$

$$= 6 \ln\left(\frac{5 + \sqrt{21}}{3 + \sqrt{5}}\right) \approx 3.626$$

84. (a) $v(t) = -32t$

(b)
$$s(t) = \int v(t) dt = \int (-32t) dt = -16t^2 + C$$

$$s(0) = -16(0)^2 + C = 400 \quad C = 400$$

$$s(t) = -16t^2 + 400$$

—CONTINUED—

84. —CONTINUED—

$$(c) \quad \frac{dv}{dt} = -32 + kv^2$$

$$\int \frac{dv}{kv^2 - 32} = \int dt$$

$$\int \frac{dv}{32 - kv^2} = - \int dt$$

$$\text{Let } u = \sqrt{k}v, \text{ then } du = \sqrt{k}dv.$$

$$\frac{1}{\sqrt{k}} \cdot \frac{1}{2\sqrt{32}} \ln \left| \frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} \right| = -t + C$$

$$\text{Since } v(0) = 0, C = 0.$$

$$\ln \left| \frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} \right| = -2\sqrt{32k}t$$

$$\frac{\sqrt{32} + \sqrt{k}v}{\sqrt{32} - \sqrt{k}v} = e^{-2\sqrt{32k}t}$$

$$\sqrt{32} + \sqrt{k}v = e^{-2\sqrt{32k}t}(\sqrt{32} - \sqrt{k}v)$$

$$v(\sqrt{k} + \sqrt{k}e^{-2\sqrt{32k}t}) = \sqrt{32}(e^{-2\sqrt{32k}t} - 1)$$

$$v = \frac{\sqrt{32}(e^{-2\sqrt{32k}t} - 1)}{\sqrt{k}(e^{-2\sqrt{32k}t} + 1)} \cdot \frac{e^{\sqrt{32k}t}}{e^{\sqrt{32k}t}}$$

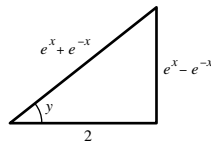
$$= \frac{\sqrt{32} \left[- (e^{\sqrt{32k}t} - e^{-\sqrt{32k}t}) \right]}{\sqrt{k} \left[e^{\sqrt{32k}t} + e^{-\sqrt{32k}t} \right]}$$

$$= -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t)$$

86. Let $y = \arcsin(\tanh x)$. Then,

$$\sin y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and}$$

$$\tan y = \frac{e^x - e^{-x}}{2} = \sinh x.$$



Thus, $y = \arctan(\sinh x)$. Therefore,

$$\arctan(\sinh x) = \arcsin(\tanh x).$$

88. $y = \operatorname{sech}^{-1} x$

$$\operatorname{sech} y = x$$

$$-(\operatorname{sech} y)(\tanh y)y' = 1$$

$$y' = \frac{-1}{(\operatorname{sech} y)(\tanh y)} = \frac{-1}{(\operatorname{sech} y)\sqrt{1 - \operatorname{sech}^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

90. $y = \sinh^{-1} x$

$$\sinh y = x$$

$$(\cosh y)y' = 1$$

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}$$

$$(d) \quad \lim_{t \rightarrow \infty} \left[-\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t) \right] = -\frac{\sqrt{32}}{\sqrt{k}}.$$

The velocity is bounded by $-\sqrt{32}/\sqrt{k}$.

(e) Since $\int \tanh(ct) dt = (1/c) \ln \cosh(ct)$ (which can be verified by differentiation), then

$$\begin{aligned} s(t) &= \int -\frac{\sqrt{32}}{\sqrt{k}} \tanh(\sqrt{32k}t) dt \\ &= -\frac{\sqrt{32}}{\sqrt{k}} \frac{1}{\sqrt{32k}} \ln[\cosh(\sqrt{32k}t)] + C \\ &= -\frac{1}{k} \ln[\cosh(\sqrt{32k}t)] + C. \end{aligned}$$

When $t = 0$,

$$s(0) = C$$

$$= 400 - 400 - (1/k) \ln[\cosh(\sqrt{32k}t)].$$

When $k = 0.01$,

$$s_2(t) = 400 - 100 \ln(\cosh \sqrt{0.32}t)$$

$$s_1(t) = -16t^2 + 400.$$

$s_1(t) = 0$ when $t = 5$ seconds.

$s_2(t) = 0$ when $t \approx 8.3$ seconds

When air resistance is not neglected, it takes approximately 3.3 more seconds to reach the ground.