

CHAPTER 2

Differentiation

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CHAPTER 2

Differentiation

Section 2.1 The Derivative and the Tangent Line Problem

Solutions to Even-Numbered Exercises

2. (a) $m = \frac{1}{4}$

(b) $m = 1$

6. $g(x) = \frac{3}{2}x + 1$ is a line. Slope = $\frac{3}{2}$

10. Slope at $(-2, 7) = \lim_{\Delta t \rightarrow 0} \frac{h(-2 + \Delta t) - h(-2)}{\Delta t}$
 $= \lim_{\Delta t \rightarrow 0} \frac{(-2 + \Delta t)^2 + 3 - 7}{\Delta t}$
 $= \lim_{\Delta t \rightarrow 0} \frac{4 - 4(\Delta t) + (\Delta t)^2 - 4}{\Delta t}$
 $= \lim_{\Delta t \rightarrow 0} (-4 + \Delta t) = -4$

14. $f(x) = 3x + 2$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{[3(x + \Delta x) + 2] - [3x + 2]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} 3 = 3$$

4. (a) $\frac{f(4) - f(1)}{4 - 1} = \frac{5 - 2}{3} = 1$

$$\frac{f(4) - f(3)}{4 - 3} \approx \frac{5 - 4.75}{1} = 0.25$$

Thus, $\frac{f(4) - f(1)}{4 - 1} > \frac{f(4) - f(3)}{4 - 3}$

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. Thus,

$$\frac{f(4) - f(1)}{4 - 1} < f'(1).$$

8. Slope at $(2, 1) = \lim_{\Delta x \rightarrow 0} \frac{g(2 + \Delta x) - g(2)}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \frac{5 - (2 + \Delta x)^2 - 1}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \frac{5 - 4 - 4(\Delta x) - (\Delta x)^2 - 1}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} (-4 - \Delta x) = -4$

12. $g(x) = -5$

$$g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-5 - (-5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

16. $f(x) = 9 - \frac{1}{2}x$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{[9 - (1/2)(x + \Delta x)] - [9 - (1/2)x]}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

18. $f(x) = 1 - x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[1 - (x + \Delta x)^2] - [1 - x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1 - x^2 - 2x\Delta x - (\Delta x)^2 - 1 + x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-2x - \Delta x) = -2x \end{aligned}$$

20. $f(x) = x^3 + x^2$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + (x + \Delta x)^2] - [x^3 + x^2]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + (\Delta x)^2 + 2x + (\Delta x)) = 3x^2 + 2x \end{aligned}$$

22. $f(x) = \frac{1}{x^2}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 - (x + \Delta x)^2}{\Delta x(x + \Delta x)^2 x^2} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{\Delta x(x + \Delta x)^2 x^2} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x - \Delta x}{(x + \Delta x)^2 x^2} \\ &= \frac{-2x}{x^4} \\ &= -\frac{2}{x^3} \end{aligned}$$

24. $f(x) = \frac{4}{\sqrt{x}}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{4}{\sqrt{x + \Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{4\sqrt{x} - 4\sqrt{x + \Delta x}}{\Delta x \sqrt{x} \sqrt{x + \Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x + \Delta x}}{\sqrt{x} + \sqrt{x + \Delta x}} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{4x - 4(x + \Delta x)}{\Delta x \sqrt{x} \sqrt{x + \Delta x} (\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-4}{\sqrt{x} \sqrt{x + \Delta x} (\sqrt{x} + \sqrt{x + \Delta x})} \\ &= \frac{-4}{\sqrt{x} \sqrt{x} (\sqrt{x} + \sqrt{x})} = \frac{-2}{x\sqrt{x}} \end{aligned}$$

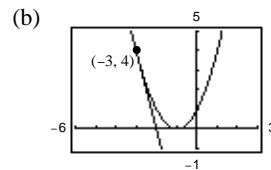
26. (a) $f(x) = x^2 + 2x + 1$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^2 + 2(x + \Delta x) + 1] - [x^2 + 2x + 1]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x + 2) = 2x + 2 \end{aligned}$$

At $(-3, 4)$, the slope of the tangent line is $m = 2(-3) + 2 = -4$.

The equation of the tangent line is

$$\begin{aligned} y - 4 &= -4(x + 3) \\ y &= -4x - 8. \end{aligned}$$



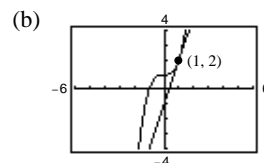
28. (a) $f(x) = x^3 + 1$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[(x + \Delta x)^3 + 1] - (x^3 + 1)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2 \end{aligned}$$

At $(1, 2)$, the slope of the tangent line is $m = 3(1)^2 = 3$.

The equation of the tangent line is

$$\begin{aligned} y - 2 &= 3(x - 1) \\ y &= 3x - 1. \end{aligned}$$



30. (a) $f(x) = \sqrt{x - 1}$

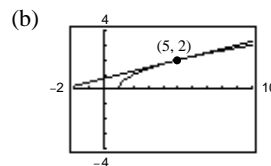
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x - 1} - \sqrt{x - 1}}{\Delta x} \cdot \left(\frac{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x - 1) - (x - 1)}{\Delta x(\sqrt{x + \Delta x - 1} + \sqrt{x - 1})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x - 1} + \sqrt{x - 1}} = \frac{1}{2\sqrt{x - 1}} \end{aligned}$$

At $(5, 2)$, the slope of the tangent line is

$$m = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{4}$$

The equation of the tangent line is

$$\begin{aligned} y - 2 &= \frac{1}{4}(x - 5) \\ y &= \frac{1}{4}x + \frac{3}{4} \end{aligned}$$



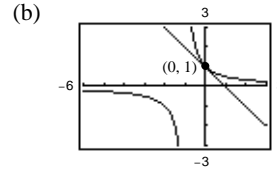
32. (a) $f(x) = \frac{1}{x+1}$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x + \Delta x + 1} - \frac{1}{x + 1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + 1) - (x + \Delta x + 1)}{\Delta x(x + \Delta x + 1)(x + 1)} \\ &= \lim_{\Delta x \rightarrow 0} -\frac{1}{(x + \Delta x + 1)(x + 1)} \\ &= -\frac{1}{(x + 1)^2} \end{aligned}$$

At $(0, 1)$, the slope of the tangent line is

$$m = \frac{-1}{(0 + 1)^2} = -1.$$

The equation of the tangent line is $y = -x + 1$.



34. Using the limit definition of derivative, $f'(x) = 3x^2$. Since the slope of the given line is 3, we have

$$3x^2 = 3$$

$$x^2 = 1 \quad x = \pm 1.$$

Therefore, at the points $(1, 3)$ and $(-1, 1)$ the tangent lines are parallel to $3x - y - 4 = 0$. These lines have equations

$$y - 3 = 3(x - 1) \text{ and } y - 1 = 3(x + 1)$$

$$y = 3x \qquad y = 3x + 4$$

36. Using the limit definition of derivative, $f'(x) = \frac{-1}{2(x-1)^{3/2}}$.

Since the slope of the given line is $-\frac{1}{2}$, we have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$1 = (x-1)^{3/2}$$

$$1 = x - 1 \quad x = 2$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2$$

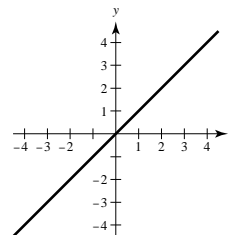
38. $h(-1) = 4$ because the tangent line passes through $(-1, 4)$

$$h'(-1) = \frac{6 - 4}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$

40. $f(x) = x^2 \quad f'(x) = 2x$ (d)

42. f' does not exist at $x = 0$. Matches (c)

44.



Answers will vary.

Sample answer: $y = x$

46. (a) Yes. $\lim_{\Delta x \rightarrow 0} \frac{f(x + 2\Delta x) - f(x)}{2\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

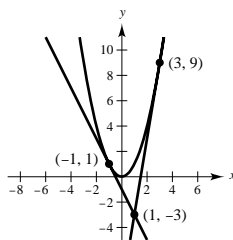
(b) No. The numerator does not approach zero.

(c) Yes.
$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x) - f(x - \Delta x) + f(x)}{2\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) - f(x)}{2\Delta x} + \frac{f(x - \Delta x) - f(x)}{2(-\Delta x)} \right] \\ &= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x) \end{aligned}$$

(d) Yes. $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$

48. Let (x_0, y_0) be a point of tangency on the graph of f . By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0 :

$$\begin{aligned} \frac{-3 - y_0}{1 - x_0} &= 2x_0 \\ -3 - y_0 &= (1 - x_0)2x_0 \\ -3 - x_0^2 &= 2x_0 - 2x_0^2 \\ x_0^2 - 2x_0 - 3 &= 0 \\ (x_0 - 3)(x_0 + 1) &= 0 \quad x_0 = 3, -1 \end{aligned}$$



Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2 . The equations of the tangent lines are

$$\begin{aligned} y + 3 &= 6(x - 1) & y + 3 &= -2(x - 1) \\ y &= 6x - 9 & y &= -2x - 1 \end{aligned}$$

50. (a) $f(x) = x^2$

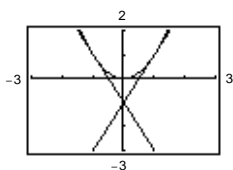
$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x \end{aligned}$$

At $x = -1$, $f'(-1) = -2$ and the tangent line is

$$y - 1 = -2(x + 1) \quad \text{or} \quad y = -2x - 1.$$

At $x = 0$, $f'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $f'(1) = 2$ and the tangent line is $y = 2x - 1$.



For this function, the slopes of the tangent lines are always distinct for different values of x .

(b) $g'(x) = \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$

$$\begin{aligned} &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x(\Delta x) + (\Delta x)^2) = 3x^2 \end{aligned}$$

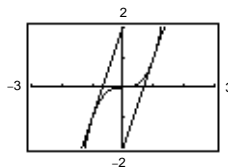
At $x = -1$, $g'(-1) = 3$ and the tangent line is

$$y + 1 = 3(x + 1) \quad \text{or} \quad y = 3x + 2.$$

At $x = 0$, $g'(0) = 0$ and the tangent line is $y = 0$.

At $x = 1$, $g'(1) = 3$ and the tangent line is

$$y - 1 = 3(x - 1) \quad \text{or} \quad y = 3x - 2.$$

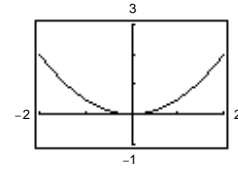


For this function, the slopes of the tangent lines are sometimes the same.

52. $f(x) = \frac{1}{2}x^2$

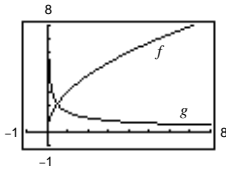
By the limit definition of the derivative we have $f'(x) = x$.

| | | | | | | | | | |
|---------|----|-------|-----|-------|---|-------|-----|-------|---|
| x | -2 | -1.5 | -1 | -0.5 | 0 | 0.5 | 1 | 1.5 | 2 |
| $f(x)$ | 2 | 1.125 | 0.5 | 0.125 | 0 | 0.125 | 0.5 | 1.125 | 2 |
| $f'(x)$ | -2 | -1.5 | -1 | -0.5 | 0 | 0.5 | 1 | 1.5 | 2 |



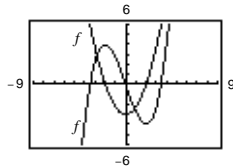
54.
$$g(x) = \frac{f(x + 0.01) - f(x)}{0.01}$$

$$= (3\sqrt{x + 0.01} - 3\sqrt{x})100$$



The graph of $g(x)$ is approximately the graph of $f'(x)$.

58. $f(x) = \frac{x^3}{4} - 3x$ and $f'(x) = \frac{3}{4}x^2 - 3$



60. $f(x) = x + \frac{1}{x}$

$$S_{\Delta x}(x) = \frac{f(2 + \Delta x) - f(2)}{\Delta x}(x - 2) + f(2) = \frac{(2 + \Delta x) + \frac{1}{2 + \Delta x} - \frac{5}{2}}{\Delta x}(x - 2) + \frac{5}{2}$$

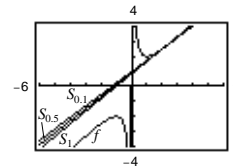
$$= \frac{2(2 + \Delta x)^2 + 2 - 5(2 + \Delta x)}{2(2 + \Delta x)\Delta x}(x - 2) + \frac{5}{2} = \frac{(2\Delta x + 3)}{2(2 + \Delta x)}(x - 2) + \frac{5}{2}$$

(a) $\Delta x = 1$: $S_{\Delta x} = \frac{5}{6}(x - 2) + \frac{5}{2} = \frac{5}{6}x + \frac{5}{6}$

$\Delta x = 0.5$: $S_{\Delta x} = \frac{4}{5}(x - 2) + \frac{5}{2} = \frac{4}{5}x + \frac{9}{10}$

$\Delta x = 0.1$: $S_{\Delta x} = \frac{16}{21}(x - 2) + \frac{5}{2} = \frac{16}{21}x + \frac{41}{42}$

(b) As $\Delta x \rightarrow 0$, the line approaches the tangent line to f at $(2, \frac{5}{2})$.



62. $g(x) = x(x - 1) = x^2 - x$, $c = 1$

$$g'(1) = \lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - x - 0}{x - 1} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{x - 1}$$

$$= \lim_{x \rightarrow 1} x = 1$$

64. $f(x) = x^3 + 2x, c = 1$

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 3)}{x - 1} = \lim_{x \rightarrow 1} (x^2 + x + 3) = 5$$

66. $f(x) = \frac{1}{x}, c = 3$

$$f'(3) = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{(1/x) - (1/3)}{x - 3} = \lim_{x \rightarrow 3} \frac{3 - x}{3x} \cdot \frac{1}{x - 3} = \lim_{x \rightarrow 3} \left(-\frac{1}{3x} \right) = -\frac{1}{9}$$

68. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x \rightarrow -3} \frac{g(x) - g(-3)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x \rightarrow -3} \frac{1}{(x + 3)^{2/3}}$$

Does not exist.

70. $f(x) = |x - 4|, c = 4$

$$f'(4) = \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4| - 0}{x - 4} = \lim_{x \rightarrow 4} \frac{|x - 4|}{x - 4}$$

Does not exist.

72. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph.)74. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)76. $f(x)$ is differentiable everywhere except at $x = 0$. (Sharp turn in the graph)78. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)80. $f(x)$ is differentiable everywhere except at $x = 1$. (Discontinuity)

82. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1} = \lim_{x \rightarrow 1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} = \lim_{x \rightarrow 1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty. \text{ (Vertical tangent)}$$

The limit from the right does not exist since f is undefined for $x > 1$. Therefore, f is not differentiable at $x = 1$.

84. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2.$$

These one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

86. Note that f is continuous at $x = 2$. $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

The derivative from the left is

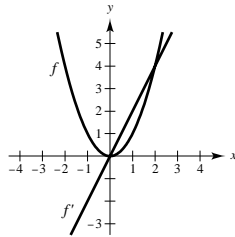
$$\lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2^-} \frac{(\frac{1}{2}x + 1) - 2}{x - 2} = \lim_{x \rightarrow 2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}$$

The derivative from the right is

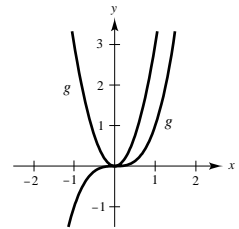
$$\begin{aligned} \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \rightarrow 2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2} \\ &= \lim_{x \rightarrow 2^+} \frac{2x - 4}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2(x - 2)}{(x - 2)(\sqrt{2x} + 2)} = \lim_{x \rightarrow 2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2} \end{aligned}$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$. ($f'(2) = \frac{1}{2}$)

88. (a) $f(x) = x^2$ and $f'(x) = 2x$



- (b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

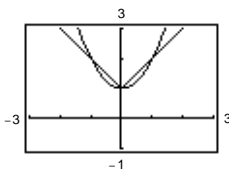
$$\begin{aligned} &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3) = 4x^3 \end{aligned}$$

Hence, if $f(x) = x^4$, then $f'(x) = 4x^3$ which is consistent with the conjecture. However, this is not a proof, since you must verify the conjecture for all integer values of n , $n \geq 2$.

90. False. $y = |x - 2|$ is continuous at $x = 2$, but is not differentiable at $x = 2$. (Sharp turn in the graph)

92. True—see Theorem 2.1

- 94.



As you zoom in, the graph of $y_1 = x^2 + 1$ appears to be locally the graph of a horizontal line, whereas the graph of $y_2 = |x| + 1$ always has a sharp corner at $(0, 1)$. y_2 is not differentiable at $(0, 1)$.

Section 2.2 Basic Differentiation Rules and Rates of Change

$$\begin{aligned} 2. \text{ (a)} \quad y &= x^{-1/2} \\ y' &= -\frac{1}{2}x^{-3/2} \\ y'(1) &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y &= x^{-1} \\ y' &= -x^{-2} \\ y'(1) &= -1 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad y &= x^{-3/2} \\ y' &= -\frac{3}{2}x^{-5/2} \\ y'(1) &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad y &= x^{-2} \\ y' &= -2x^{-3} \\ y'(1) &= -2 \end{aligned}$$

$$\begin{aligned} 4. \quad f(x) &= -2 \\ f'(x) &= 0 \end{aligned}$$

$$\begin{aligned} 6. \quad y &= x^8 \\ y' &= 8x^7 \end{aligned}$$

$$\begin{aligned} 8. \quad y &= \frac{1}{x^8} = x^{-8} \\ y' &= 8x^{-9} = \frac{-8}{x^9} \end{aligned}$$

$$\begin{aligned} 10. \quad y &= \sqrt[4]{x} = x^{1/4} \\ y' &= \frac{1}{4}x^{-3/4} = \frac{1}{4x^{3/4}} \end{aligned}$$

$$\begin{aligned} 12. \quad g(x) &= 3x - 1 \\ g'(x) &= 3 \end{aligned}$$

$$\begin{aligned} 14. \quad y &= t^2 + 2t - 3 \\ y' &= 2t + 2 \end{aligned}$$

$$\begin{aligned} 16. \quad y &= 8 - x^3 \\ y' &= -3x^2 \end{aligned}$$

$$\begin{aligned} 18. \quad f(x) &= 2x^3 - x^2 + 3x \\ f'(x) &= 6x^2 - 2x + 3 \end{aligned}$$

$$\begin{aligned} 20. \quad g(t) &= \pi \cos t \\ g'(t) &= -\pi \sin t \end{aligned}$$

$$\begin{aligned} 22. \quad y &= 5 + \sin x \\ y' &= \cos x \end{aligned}$$

$$\begin{aligned} 24. \quad y &= \frac{5}{(2x)^3} + 2 \cos x = \frac{5}{8}x^{-3} + 2 \cos x \\ y' &= \frac{5}{8}(-3)x^{-4} - 2 \sin x = \frac{-15}{8x^4} - 2 \sin x \end{aligned}$$

| <i>Function</i> | <i>Rewrite</i> | <i>Derivative</i> | <i>Simplify</i> |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------|---------------------------|
| 26. $y = \frac{2}{3x^2}$ | $y = \frac{2}{3}x^{-2}$ | $y' = -\frac{4}{3}x^{-3}$ | $y' = -\frac{4}{3x^3}$ |
| 28. $y = \frac{\pi}{(3x)^2}$ | $y = \frac{\pi}{9}x^{-2}$ | $y' = -\frac{2\pi}{9}x^{-3}$ | $y' = -\frac{2\pi}{9x^3}$ |
| 30. $y = \frac{4}{x^{-3}}$ | $y = 4x^3$ | $y' = 12x^2$ | $y' = 12x^2$ |
| 32. $f(t) = 3 - \frac{3}{5t} \left(\frac{3}{5}, 2\right)$ $f'(t) = \frac{3}{5t^2}$ $f'\left(\frac{3}{5}\right) = \frac{5}{3}$ | 34. $y = 3x^3 - 6, (2, 18)$ $y' = 9x^2$ $y'(2) = 36$ | 36. $f(x) = 3(5 - x)^2, (5, 0)$ $= 3x^2 - 30x + 75$ $f'(x) = 6x - 30$ $f'(5) = 0$ | |
| 38. $g(t) = 2 + 3 \cos t, (\pi, -1)$ $g'(t) = -3 \sin t$ $g'(\pi) = 0$ | 40. $f(x) = x^2 - 3x - 3x^{-2}$ $f'(x) = 2x - 3 + 6x^{-3}$ $= 2x - 3 + \frac{6}{x^3}$ | 42. $f(x) = x + x^{-2}$ $f'(x) = 1 - 2x^{-3}$ $= 1 - \frac{2}{x^3}$ | |
| 44. $h(x) = \frac{2x^2 - 3x + 1}{x} = 2x - 3 + x^{-1}$ $h'(x) = 2 - \frac{1}{x^2} = \frac{2x^2 - 1}{x^2}$ | 46. $y = 3x(6x - 5x^2) = 18x^2 - 15x^3$ $y' = 36x - 45x^2$ | | |
| 48. $f(x) = \sqrt[3]{x} + \sqrt[5]{x} = x^{1/3} + x^{1/5}$ $f'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5} = \frac{1}{3x^{2/3}} + \frac{1}{5x^{4/5}}$ | 50. $f(t) = t^{2/3} - t^{1/3} + 4$ $f'(t) = \frac{2}{3}t^{-1/3} - \frac{1}{3}t^{-2/3} = \frac{2}{3t^{1/3}} - \frac{1}{3t^{2/3}}$ | | |

$$52. f(x) = \frac{2}{\sqrt[3]{x}} + 3 \cos x = 2x^{-1/3} + 3 \cos x$$

$$f'(x) = \frac{-2}{3}x^{-4/3} - 3 \sin x = \frac{-2}{3x^{4/3}} - 3 \sin x$$

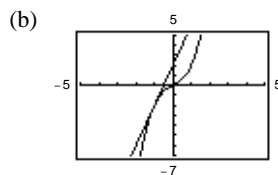
$$54. (a) y = x^3 + x$$

$$y' = 3x^2 + 1$$

$$\text{At } (-1, -2): y' = 3(-1)^2 + 1 = 4.$$

$$\text{Tangent line: } y + 2 = 4(x + 1)$$

$$4x - y + 2 = 0$$



$$56. (a) y = (x^2 + 2x)(x + 1)$$

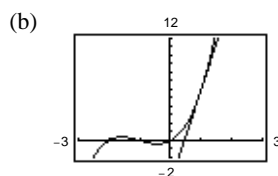
$$= x^3 + 3x^2 + 2x$$

$$y' = 3x^2 + 6x + 2$$

$$\text{At } (1, 6): y' = 3(1)^2 + 6(1) + 2 = 11.$$

$$\text{Tangent line: } y - 6 = 11(x - 1)$$

$$0 = 11x - y - 5$$



$$58. y = x^3 + x$$

$$y' = 3x^2 + 1 > 0 \text{ for all } x.$$

Therefore, there are no horizontal tangents.

$$60. y = x^2 + 1$$

$$y' = 2x = 0 \quad x = 0$$

$$\text{At } x = 0, y = 1.$$

Horizontal tangent: (0, 1)

$$62. y = \sqrt{3}x + 2 \cos x, \quad 0 < x < 2\pi$$

$$y' = \sqrt{3} - 2 \sin x = 0$$

$$\sin x = \frac{\sqrt{3}}{2} \quad x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$$\text{At } x = \frac{\pi}{3}, y = \frac{\sqrt{3}\pi + 3}{3}.$$

$$\text{At } x = \frac{2\pi}{3}, y = \frac{2\sqrt{3}\pi - 3}{3}.$$

$$\text{Horizontal tangents: } \left(\frac{\pi}{3}, \frac{\sqrt{3}\pi + 3}{3}\right), \left(\frac{2\pi}{3}, \frac{2\sqrt{3}\pi - 3}{3}\right)$$

$$64. k - x^2 = -4x + 7 \quad \text{Equate functions}$$

$$-2x = -4 \quad \text{Equate derivatives}$$

$$\text{Hence, } x = 2 \text{ and } k - 4 = -8 + 7 \quad k = 3$$

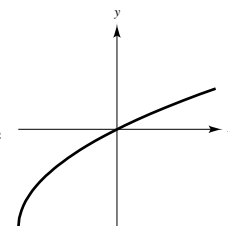
$$66. k\sqrt{x} = x + 4 \quad \text{Equate functions}$$

$$\frac{k}{2\sqrt{x}} = 1 \quad \text{Equate derivatives}$$

$$\text{Hence, } k = 2\sqrt{x} \text{ and}$$

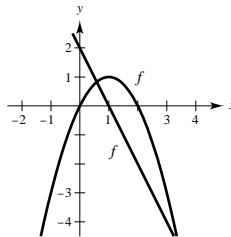
$$(2\sqrt{x})\sqrt{x} = x + 4 \quad 2x = x + 4 \quad x = 4 \quad k = 4$$

68. The graph of a function f such that $f' > 0$ for all x and the rate of change the function is decreasing (i.e. $f'' < 0$) would, in general, look like the graph at the right.



70. $g(x) = -5f(x) \quad g'(x) = -5f'(x)$

72.



If f is quadratic, then its derivative is a linear function.

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b$$

74. m_1 is the slope of the line tangent to $y = x$. m_2 is the slope of the line tangent to $y = 1/x$. Since

$$y = x \quad y' = 1 \quad m_1 = 1 \quad \text{and} \quad y = \frac{1}{x} \quad y' = \frac{-1}{x^2} \quad m_2 = \frac{-1}{x^2}.$$

The points of intersection of $y = x$ and $y = 1/x$ are

$$x = \frac{1}{x} \quad x^2 = 1 \quad x = \pm 1.$$

At $x = \pm 1$, $m_2 = -1$. Since $m_2 = -1/m_1$, these tangent lines are perpendicular at the points intersection.

76. $f(x) = \frac{2}{x}, (5, 0)$

78. $f'(4) = 1$

$$f'(x) = -\frac{2}{x^2}$$

$$-\frac{2}{x^2} = \frac{0 - y}{5 - x}$$

$$-10 + 2x = -x^2y$$

$$-10 + 2x = -x^2\left(\frac{2}{x}\right)$$

$$-10 + 2x = -2x$$

$$4x = 10$$

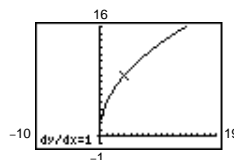
$$x = \frac{5}{2}, y = \frac{4}{5}$$

The point $\left(\frac{5}{2}, \frac{4}{5}\right)$ is on the graph of f . The slope of the tangent line is $f'\left(\frac{5}{2}\right) = -\frac{8}{25}$.

Tangent line: $y - \frac{4}{5} = -\frac{8}{25}\left(x - \frac{5}{2}\right)$

$$25y - 20 = -8x + 20$$

$$8x + 25y - 40 = 0$$

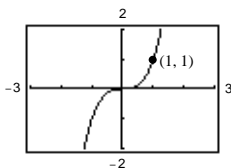


80. (a) Nearby point: (1.0073138, 1.0221024)

$$\text{Secant line: } y - 1 = \frac{1.0221024 - 1}{1.0073138 - 1}(x - 1)$$

$$y = 3.022(x - 1) + 1$$

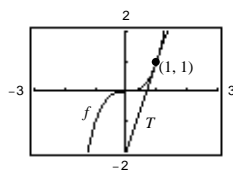
(Answers will vary.)



(b) $f'(x) = 3x^2$

$$T(x) = 3(x - 1) + 1 = 3x - 2$$

(c) The accuracy worsens as you move away from (1, 1).



(d)

| | | | | | | | | | | | |
|------------|----|----|----|-------|-------|---|-------|-------|---|----|----|
| Δx | -3 | -2 | -1 | -0.5 | -0.1 | 0 | 0.1 | 0.5 | 1 | 2 | 3 |
| $f(x)$ | -8 | -1 | 0 | 0.125 | 0.729 | 1 | 1.331 | 3.375 | 8 | 27 | 64 |
| $T(x)$ | -8 | -5 | -2 | -0.5 | 0.7 | 1 | 1.3 | 2.5 | 4 | 7 | 10 |

The accuracy decreases more rapidly than in Exercise 59 because $y = x^3$ is less “linear” than $y = x^{3/2}$.

82. True. If $f(x) = g(x) + c$, then $f'(x) = g'(x) + 0 = g'(x)$.

84. True. If $y = x/\pi = (1/\pi) \cdot x$, then $dy/dx = (1/\pi)(1) = 1/\pi$.

86. False. If $f(x) = \frac{1}{x^n} = x^{-n}$, then $f'(x) = -nx^{-n-1} = \frac{-n}{x^{n+1}}$

88. $f(t) = t^2 - 3, [2, 2.1]$

$$f'(t) = 2t$$

Instantaneous rate of change:

$$(2, 1) \quad f'(2) = 2(2) = 4$$

$$(2.1, 1.41) \quad f'(2.1) = 4.2$$

Average rate of change:

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{1.41 - 1}{0.1} = 4.1$$

90. $f(x) = \sin x, \left[0, \frac{\pi}{6}\right]$

$$f'(x) = \cos x$$

Instantaneous rate of change:

$$(0, 0) \quad f'(0) = 1$$

$$\left(\frac{\pi}{6}, \frac{1}{2}\right) \quad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$$

Average rate of change:

$$\frac{f(\pi/6) - f(0)}{(\pi/6) - 0} = \frac{(1/2) - 0}{(\pi/6) - 0} = \frac{3}{\pi} \approx 0.955$$

92.

$$s(t) = -16t^2 - 22t + 220$$

$$v(t) = -32t - 22$$

$$v(3) = -118 \text{ ft/sec}$$

$$s(t) = -16t^2 - 22t + 220$$

$$= 112 \text{ (height after falling 108 ft)}$$

$$-16t^2 - 22t + 108 = 0$$

$$-2(t - 2)(8t + 27) = 0$$

$$t = 2$$

$$v(2) = -32(2) - 22$$

$$= -86 \text{ ft/sec}$$

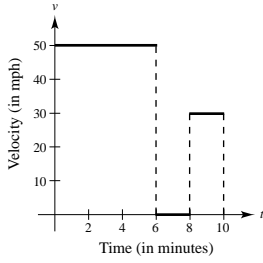
94.

$$s(t) = -4.9t^2 + v_0t + s_0$$

$$= -4.9t^2 + s_0 = 0 \text{ when } t = 6.8.$$

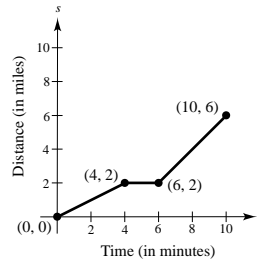
$$s_0 = 4.9t^2 = 4.9(6.8)^2 = 226.6 \text{ m}$$

96.



(The velocity has been converted to miles per hour)

98. This graph corresponds with Exercise 75.



100. $s(t) = -\frac{1}{2}at^2 + c$ and $s'(t) = -at$.

$$\begin{aligned} \text{Average velocity: } \frac{s(t_0 + \Delta t) - s(t_0 - \Delta t)}{(t_0 + \Delta t) - (t_0 - \Delta t)} &= \frac{[-(1/2)a(t_0 + \Delta t)^2 + c] - [-(1/2)a(t_0 - \Delta t)^2 + c]}{2\Delta t} \\ &= \frac{-(1/2)a(t_0^2 + 2t_0\Delta t + (\Delta t)^2) + (1/2)a(t_0^2 - 2t_0\Delta t + (\Delta t)^2)}{2\Delta t} \\ &= \frac{-2at_0\Delta t}{2\Delta t} \\ &= -at_0 \\ &= s'(t_0) \text{ Instantaneous velocity at } t = t_0 \end{aligned}$$

102. $V = s^3, \frac{dV}{ds} = 3s^2$

When $s = 4$ cm, $\frac{dV}{ds} = 48 \text{ cm}^2$.

104. $C = (\text{gallons of fuel used})(\text{cost per gallon})$

$$= \left(\frac{15,000}{x}\right)(1.25) = \frac{18,750}{x}$$

$$\frac{dC}{dx} = -\frac{18,750}{x^2}$$

| | | | | | | | |
|-----------------|--------|---------|---------|-----|---------|---------|---------|
| x | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| C | 1875 | 1250 | 537.5 | 750 | 625 | 535.71 | 468.75 |
| $\frac{dC}{dx}$ | -187.5 | -83.333 | -46.875 | -30 | -20.833 | -15.306 | -11.719 |

The driver who gets 15 miles per gallon would benefit more from a 1 mile per gallon increase in fuel efficiency. The rate of change is larger when $x = 15$.

106. $\frac{dT}{dt} = K(T - T_a)$

108. $y = \frac{1}{x}, x > 0$

$$y' = -\frac{1}{x^2}$$

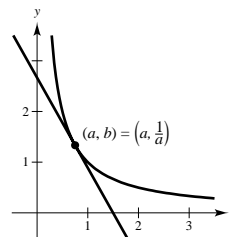
At (a, b) , the equation of the tangent line is

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \quad \text{or} \quad y = -\frac{x}{a^2} + \frac{2}{a}$$

The x -intercept is $(2a, 0)$.

The y -intercept is $(0, \frac{2}{a})$.

The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2a)(\frac{2}{a}) = 2$.



110. $y = x^2$

$$y' = 2x$$

(a) Tangent lines through $(0, a)$:

$$y - a = 2x(x - 0)$$

$$x^2 - a = 2x^2$$

$$-a = x^2$$

$$\pm\sqrt{-a} = x$$

The points of tangency are $(\pm\sqrt{-a}, -a)$. At $(\sqrt{-a}, -a)$ the slope is $y'(\sqrt{-a}) = 2\sqrt{-a}$. At $(-\sqrt{-a}, -a)$ the slope is $y'(-\sqrt{-a}) = -2\sqrt{-a}$.

Tangent lines: $y + a = 2\sqrt{-a}(x - \sqrt{-a})$ and $y + a = -2\sqrt{-a}(x + \sqrt{-a})$

$$y = 2\sqrt{-a}x + a$$

$$y = -2\sqrt{-a}x + a$$

Restriction: a must be negative.

(b) Tangent lines through $(a, 0)$:

$$y - 0 = 2x(x - a)$$

$$x^2 = 2x^2 - 2ax$$

$$0 = x^2 - 2ax = x(x - 2a)$$

The points of tangency are $(0, 0)$ and $(2a, 4a^2)$. At $(0, 0)$ the slope is $y'(0) = 0$. At $(2a, 4a^2)$ the slope is $y'(2a) = 4a$.

Tangent lines: $y - 0 = 0(x - 0)$ and $y - 4a^2 = 4a(x - 2a)$

$$y = 0$$

$$y = 4ax - 4a^2$$

Restriction: None, a can be any real number.

112. $f_1(x) = |\sin x|$ is differentiable for all $x \neq n\pi$, n an integer.

$f_2(x) = \sin|x|$ is differentiable for all $x \neq 0$.

You can verify this by graphing f_1 and f_2 and observing the locations of the sharp turns.

Section 2.3 The Product and Quotient Rules and Higher-Order Derivatives

2. $f(x) = (6x + 5)(x^3 - 2)$

$$\begin{aligned} f'(x) &= (6x + 5)(3x^2) + (x^3 - 2)(6) \\ &= 18x^3 + 15x^2 + 6x^3 - 12 \\ &= 24x^3 + 15x^2 - 12 \end{aligned}$$

6. $g(x) = \sqrt{x} \sin x$

$$g'(x) = \sqrt{x} \cos x + \sin x \left(\frac{1}{2\sqrt{x}} \right) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$$

10. $h(s) = \frac{s}{\sqrt{s} - 1}$

$$\begin{aligned} h'(s) &= \frac{(\sqrt{s} - 1)(1) - s\left(\frac{1}{2}s^{-1/2}\right)}{(\sqrt{s} - 1)^2} \\ &= \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{(\sqrt{s} - 1)^2} = \frac{\sqrt{s} - 2}{2(\sqrt{s} - 1)^2} \end{aligned}$$

14. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$\begin{aligned} f'(x) &= (x^2 - 2x + 1)(3x^2) + (x^3 - 1)(2x - 2) \\ &= 3x^2(x - 1)^2 + 2(x - 1)^2(x^2 + x + 1) \\ &= (x - 1)^2(5x^2 + 2x + 2) \\ f'(1) &= 0 \end{aligned}$$

18. $f(x) = \frac{\sin x}{x}$

$$\begin{aligned} f'(x) &= \frac{(x)(\cos x) - (\sin x)(1)}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2} \\ f'\left(\frac{\pi}{6}\right) &= \frac{(\pi/6)(\sqrt{3}/2) - (1/2)}{\pi^2/36} \\ &= \frac{3\sqrt{3}\pi - 18}{\pi^2} \\ &= \frac{3(\sqrt{3}\pi - 6)}{\pi^2} \end{aligned}$$

4. $g(s) = \sqrt{s}(4 - s^2) = s^{1/2}(4 - s^2)$

$$\begin{aligned} g'(s) &= s^{1/2}(-2s) + (4 - s^2)\frac{1}{2}s^{-1/2} = -2s^{3/2} + \frac{4 - s^2}{2s^{1/2}} \\ &= \frac{4 - 5s^2}{2s^{1/2}} \end{aligned}$$

8. $g(t) = \frac{t^2 + 2}{2t - 7}$

$$g'(t) = \frac{(2t - 7)(2t) - (t^2 + 2)(2)}{(2t - 7)^2} = \frac{2t^2 - 14t - 4}{(2t - 7)^2}$$

12. $f(t) = \frac{\cos t}{t^3}$

$$f'(t) = \frac{t^3(-\sin t) - \cos t(3t^2)}{(t^3)^2} = -\frac{t \sin t + 3 \cos t}{t^4}$$

16. $f(x) = \frac{x + 1}{x - 1}$

$$\begin{aligned} f'(x) &= \frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} \\ &= \frac{x - 1 - x - 1}{(x - 1)^2} \\ &= -\frac{2}{(x - 1)^2} \\ f'(2) &= -\frac{2}{(2 - 1)^2} = -2 \end{aligned}$$

FunctionRewriteDerivativeSimplify

20. $y = \frac{5x^2 - 3}{4}$

$y = \frac{5}{4}x^2 - \frac{3}{4}$

$y' = \frac{10}{4}x$

$y' = \frac{5x}{2}$

| <i>Function</i> | <i>Rewrite</i> | <i>Derivative</i> | <i>Simplify</i> |
|----------------------------------------------------------------------------------------------------------------------------|------------------------------------|---------------------------|------------------------|
| 22. $y = \frac{4}{5x^2}$ | $y = \frac{4}{5}x^{-2}$ | $y' = -\frac{8}{5}x^{-3}$ | $y' = -\frac{8}{5x^3}$ |
| 24. $y = \frac{3x^2 - 5}{7}$ | $y = \frac{3}{7}x^2 - \frac{5}{7}$ | $y' = \frac{6x}{7}$ | $y' = \frac{6}{7}x$ |
| 26. $f(x) = \frac{x^3 + 3x + 2}{x^2 - 1}$ | | | |
| $f'(x) = \frac{(x^2 - 1)(3x^2 + 3) - (x^3 + 3x + 2)(2x)}{(x^2 - 1)^2}$ | | | |
| $= \frac{x^4 - 6x^2 - 4x - 3}{(x^2 - 1)^2}$ | | | |
| 28. $f(x) = x^4 \left[1 - \frac{2}{x+1} \right] = x^4 \left[\frac{x-1}{x+1} \right]$ | | | |
| $f'(x) = x^4 \left[\frac{(x+1) - (x-1)}{(x+1)^2} \right] + \left[\frac{x-1}{x+1} \right] (4x^3)$ | | | |
| $= 2x^3 \left[\frac{2x^2 + x - 2}{(x+1)^2} \right]$ | | | |
| 30. $f(x) = \sqrt[3]{x}(\sqrt{x} + 3) = x^{1/3}(x^{1/2} + 3)$ | | | |
| $f'(x) = x^{1/3} \left(\frac{1}{2}x^{-1/2} \right) + (x^{1/2} + 3) \left(\frac{1}{3}x^{-2/3} \right)$ | | | |
| $= \frac{5}{6}x^{-1/6} + x^{-2/3}$ | | | |
| $= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$ | | | |
| Alternate solution: | | | |
| $f(x) = \sqrt[3]{x}(\sqrt{x} + 3)$ | | | |
| $= x^{5/6} + 3x^{1/3}$ | | | |
| $f'(x) = \frac{5}{6}x^{-1/6} + x^{-2/3}$ | | | |
| $= \frac{5}{6x^{1/6}} + \frac{1}{x^{2/3}}$ | | | |
| 32. $h(x) = (x^2 - 1)^2 = x^4 - 2x^2 + 1$ | | | |
| $h'(x) = 4x^3 - 4x = 4x(x^2 - 1)$ | | | |
| 34. $g(x) = x^2 \left(\frac{2}{x} - \frac{1}{x+1} \right) = 2x - \frac{x^2}{x+1}$ | | | |
| $g'(x) = 2 - \frac{(x+1)2x - x^2(1)}{(x+1)^2} = \frac{2(x^2 + 2x + 1) - x^2 - 2x}{(x+1)^2} = \frac{x^2 + 2x + 2}{(x+1)^2}$ | | | |
| 36. $f(x) = (x^2 - x)(x^2 + 1)(x^2 + x + 1)$ | | | |
| $f'(x) = (2x - 1)(x^2 + 1)(x^2 + x + 1) + (x^2 - x)(2x)(x^2 + x + 1) + (x^2 - x)(x^2 + 1)(2x + 1)$ | | | |
| $= (2x - 1)(x^4 + x^3 + 2x^2 + x + 1) + (x^2 - x)(2x^3 + 2x^2 + 2x) + (x^2 - x)(2x^3 + x^2 + 2x + 1)$ | | | |
| $= 2x^5 + x^4 + 3x^3 + x - 1 + 2x^5 - 2x^2 + 2x^5 - x^4 + x^3 - x^2 - x$ | | | |
| $= 6x^5 + 4x^3 - 3x^2 - 1$ | | | |
| 38. $f(x) = \frac{c^2 - x^2}{c^2 + x^2}$ | | | |
| $f'(x) = \frac{(c^2 + x^2)(-2x) - (c^2 - x^2)(2x)}{(c^2 + x^2)^2}$ | | | |
| $= \frac{-4xc^2}{(c^2 + x^2)^2}$ | | | |
| 40. $f(\theta) = (\theta + 1) \cos \theta$ | | | |
| $f'(\theta) = (\theta + 1)(-\sin \theta) + (\cos \theta)(1)$ | | | |
| $= \cos \theta - (\theta + 1) \sin \theta$ | | | |

42. $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x \cos x - \sin x}{x^2}$$

46. $h(s) = \frac{1}{s} - 10 \csc s$

$$h'(s) = -\frac{1}{s^2} + 10 \csc s \cot s$$

50. $y = x \sin x + \cos x$

$$y' = x \cos x + \sin x - \sin x = x \cos x$$

54. $h(\theta) = 5\theta \sec \theta + \theta \tan \theta$

$$h'(\theta) = 5\theta \sec \theta \tan \theta + 5 \sec \theta + \theta \sec^2 \theta + \tan \theta$$

58. $f(\theta) = \frac{\sin \theta}{1 - \cos \theta}$

$$f'(\theta) = \frac{1}{\cos \theta - 1} = \frac{\cos \theta - 1}{(1 - \cos \theta)^2}$$

(form of answer may vary)

62. $f(x) = \sin x(\sin x + \cos x)$

$$\begin{aligned} f'(x) &= \sin x(\cos x - \sin x) + (\sin x + \cos x)\cos x \\ &= \sin x \cos x - \sin^2 x + \sin x \cos x + \cos^2 x \\ &= \sin 2x + \cos 2x \end{aligned}$$

$$f'\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1$$

64. (a) $f(x) = (x - 1)(x^2 - 2)$, $(0, 2)$

$$f'(x) = (x - 1)(2x) + (x^2 - 2)(1) = 3x^2 - 2x - 2$$

$$f'(0) = -2 = \text{slope at } (0, 2).$$

$$\text{Tangent line: } y - 2 = -2x \quad y = -2x + 2$$

66. (a) $f(x) = \frac{x - 1}{x + 1}$, $\left(2, \frac{1}{3}\right)$

$$f'(x) = \frac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2} = \frac{2}{(x + 1)^2}$$

$$f'(2) = \frac{2}{9} = \text{slope at } \left(2, \frac{1}{3}\right).$$

$$\text{Tangent line: } y - \frac{1}{3} = \frac{2}{9}(x - 2) \quad y = \frac{2}{9}x - \frac{1}{9}$$

44. $y = x + \cot x$

$$y' = 1 - \csc^2 x = -\cot^2 x$$

48. $y = \frac{\sec x}{x}$

$$y' = \frac{x \sec x \tan x - \sec x}{x^2}$$

$$= \frac{\sec x(x \tan x - 1)}{x^2}$$

52. $f(x) = \sin x \cos x$

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos 2x$$

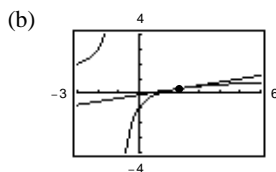
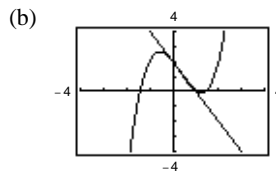
56. $f(x) = \left(\frac{x^2 - x - 3}{x^2 + 1}\right)(x^2 + x + 1)$

$$f'(x) = 2 \frac{x^5 + 2x^3 + 2x^2 - 2}{(x^2 + 1)^2} \quad (\text{form of answer may vary})$$

60. $f(x) = \tan x \cot x = 1$

$$f'(x) = 0$$

$$f'(1) = 0$$



68. (a) $f(x) = \sec x, \left(\frac{\pi}{3}, 2\right)$

$$f'(x) = \sec x \tan x$$

$$f'\left(\frac{\pi}{3}\right) = 2\sqrt{3} = \text{slope at } \left(\frac{\pi}{3}, 2\right).$$

Tangent line:

$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$6\sqrt{3}x - 3y + 6 - 2\sqrt{3}\pi = 0$$

70. $f(x) = \frac{x^2}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2)(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$$

$$f'(x) = 0 \text{ when } x = 0.$$

Horizontal tangent is at $(0, 0)$.

74. $f(x) = \frac{\cos x}{x^n} = x^{-n} \cos x$

$$f'(x) = -x^{-n} \sin x - nx^{-n-1} \cos x$$

$$= -x^{-n-1}(x \sin x + n \cos x)$$

$$= -\frac{x \sin x + n \cos x}{x^{n+1}}$$

When $n = 1$: $f'(x) = -\frac{x \sin x + \cos x}{x^2}$.

When $n = 2$: $f'(x) = -\frac{x \sin x + 2 \cos x}{x^3}$.

When $n = 3$: $f'(x) = -\frac{x \sin x + 3 \cos x}{x^4}$.

When $n = 4$: $f'(x) = -\frac{x \sin x + 4 \cos x}{x^5}$.

For general n , $f'(x) = -\frac{x \sin x + n \cos x}{x^{n+1}}$.

80. $f(x) = \sec x$

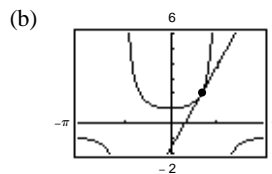
$$g(x) = \csc x, [0, 2\pi)$$

$$f'(x) = g'(x)$$

$$\sec x \tan x = -\csc x \cot x \quad \frac{\sec x \tan x}{\csc x \cot x} = -1 \quad \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = -1$$

$$\frac{\sin^3 x}{\cos^3 x} = -1 \quad \tan^3 x = -1 \quad \tan x = -1$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$



72. $f'(x) = \frac{x(\cos x - 3) - (\sin x - 3x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$

$$g'(x) = \frac{x(\cos x + 2) - (\sin x + 2x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

$$g(x) = \frac{\sin x + 2x}{x} = \frac{\sin x - 3x + 5x}{x} = f(x) + 5$$

f and g differ by a constant.

76. $V = \pi r^2 h = \pi(t + 2)\left(\frac{1}{2}\sqrt{t}\right)$

$$= \frac{1}{2}(t^{3/2} + 2t^{1/2})\pi$$

$$V'(t) = \frac{1}{2}\left(\frac{3}{2}t^{1/2} + t^{-1/2}\right)\pi = \frac{3t + 2}{4t^{1/2}}\pi \text{ cubic inches/sec}$$

78. $P = \frac{k}{V}$

$$\frac{dP}{dV} = -\frac{k}{V^2}$$

82. (a) $n(t) = -9.6643t^2 + 90.7414t + 77.5029$
 $v(t) = -276.4643t^2 + 2987.6929t + 1809.9714$

(b) $A = \frac{v(t)}{n(t)} \approx \frac{-276.46t^2 + 2987.69t + 1809.97}{-9.66t^2 + 90.74t + 77.50}$

A represents the average retail value (in millions of dollars) per 1000 motor homes.

(c) $A'(t) \approx \frac{40.46(x^2 - 2.09x + 17.83)}{(x^2 - 9.39x - 8.02)^2}$

86. $f(x) = \frac{x^2 + 2x - 1}{x} = x + 2 - \frac{1}{x}$

$f'(x) = 1 + \frac{1}{x^2}$

$f''(x) = -\frac{2}{x^3}$

90. $f''(x) = 2 - 2x^{-1}$

$f'''(x) = 2x^{-2} = \frac{2}{x^2}$

92. $f^{(4)}(x) = 2x + 1$

$f^{(5)}(x) = 2$

$f^{(6)}(x) = 0$

96. $f(x) = 4 - h(x)$

$f'(x) = -h'(x)$

$f'(2) = -h'(2) = -4$

98. $f(x) = g(x)h(x)$

$f'(x) = g(x)h'(x) + h(x)g'(x)$

$f'(2) = g(2)h'(2) + h(2)g'(2)$

$= (3)(4) + (-1)(-2)$

$= 14$

102. $s(t) = -8.25t^2 + 66t$

$v(t) = -16.50t + 66$

$a(t) = -16.50$

| | | | | | |
|---------------------------------------|-------|-------|-------|--------|-------|
| $t(\text{sec})$ | 0 | 1 | 2 | 3 | 4 |
| $s(t)$ (ft) | 0 | 57.75 | 99 | 123.75 | 132 |
| $v(t) = s'(t)$ (ft/sec) | 66 | 49.5 | 33 | 16.5 | 0 |
| $a(t) = v'(t)$ (ft/sec ²) | -16.5 | -16.5 | -16.5 | -16.5 | -16.5 |

84. $f(x) = x + \frac{32}{x^2}$

$f'(x) = 1 - \frac{64}{x^3}$

$f''(x) = \frac{192}{x^4}$

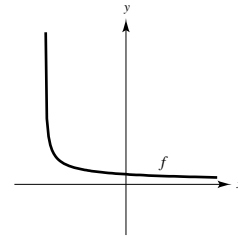
88. $f(x) = \sec x$

$f'(x) = \sec x \tan x$

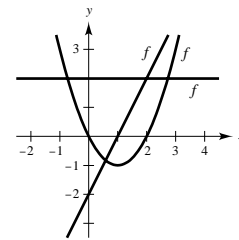
$f''(x) = \sec x(\sec^2 x) + \tan x(\sec x \tan x)$

$= \sec x(\sec^2 x + \tan^2 x)$

94. The graph of a differentiable function f such that $f > 0$ and $f' < 0$ for all real numbers x would in general look like the graph below.



100.



It appears that f is quadratic; so f' would be linear and f'' would be constant.

Average velocity on:

$[0, 1]$ is $\frac{57.75 - 0}{1 - 0} = 57.75$.

$[1, 2]$ is $\frac{99 - 57.75}{2 - 1} = 41.25$.

$[2, 3]$ is $\frac{123.75 - 99}{3 - 2} = 24.75$.

$[3, 4]$ is $\frac{132 - 123.75}{4 - 3} = 8.25$.

104. (a) $f(x) = x^n$
 $f^n(x) = n(n-1)(n-2)\cdots(2)(1) = n!$

(b) $f(x) = \frac{1}{x}$
 $f^{(n)}(x) = \frac{(-1)^n(n)(n-1)(n-2)\cdots(2)(1)}{x^{n+1}}$
 $= \frac{(-1)^n n!}{x^{n+1}}$

Note: $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (read “ n factorial.”)

106. $[xf(x)]' = xf'(x) + f(x)$
 $[xf(x)]'' = xf''(x) + f'(x) + f'(x) = xf''(x) + 2f'(x)$
 $[xf(x)]''' = xf'''(x) + f''(x) + 2f''(x) = xf'''(x) + 3f''(x)$
 In general, $[xf(x)]^{(n)} = xf^{(n)}(x) + nf^{(n-1)}(x)$.

108. $f(x) = \sin x \quad f\left(\frac{\pi}{2}\right) = 1$
 $f'(x) = \cos x \quad f'\left(\frac{\pi}{2}\right) = 0$
 $f''(x) = -\sin x \quad f''\left(\frac{\pi}{2}\right) = -1$

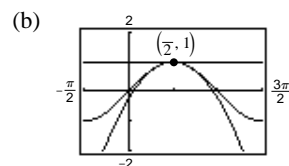
(a) $P_1(x) = f'(a)(x-a) + f(a) = 0\left(x - \frac{\pi}{2}\right) + 1 = 1$

$$P_2(x) = \frac{1}{2}f''(a)(x-a)^2 + f'(a)(x-a) + f(a) = \frac{1}{2}(-1)\left(x - \frac{\pi}{2}\right)^2 + 1$$

$$= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2$$

(c) P_2 is a better approximation than P_1 .

(d) The accuracy worsens as you move farther away from $x = a = \frac{\pi}{2}$.



110. True. y is a fourth-degree polynomial.

$$\frac{d^n y}{dx^n} = 0 \text{ when } n > 4.$$

112. True

114. True. If $v(t) = c$ then

$$a(t) = v'(t) = 0.$$

116. (a) $(fg' - f'g)' = fg'' + f'g' - f'g' - f''g$
 $= fg'' - f''g \quad \text{True}$

(b) $(fg)'' = (fg' + f'g)'$
 $= fg'' + f'g' + f'g' + f''g$
 $= fg'' + 2f'g' + f''g$
 $\neq fg'' + f''g \quad \text{False}$

Section 2.4 The Chain Rule

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

2. $y = \frac{1}{\sqrt{x+1}}$

$u = x + 1$

$y = u^{-1/2}$

4. $y = 3 \tan(\pi x^2)$

$u = \pi x^2$

$y = 3 \tan u$

6. $y = \cos \frac{3x}{2}$

$u = \frac{3x}{2}$

$y = \cos u$

8. $y = (2x^3 + 1)^2$

$y' = 2(2x^3 + 1)(6x^2) = 12x^2(2x^3 + 1)$

10. $y = 3(4 - x^2)^5$

$y' = 15(4 - x^2)(-2x) = -30x(4 - x^2)$

12. $f(t) = (9t + 2)^{2/3}$

$f'(t) = \frac{2}{3}(9t + 2)^{-1/3}(9) = \frac{6}{\sqrt[3]{9t + 2}}$

14. $g(x) = \sqrt{5 - 3x} = (5 - 3x)^{1/2}$

$g'(x) = \frac{1}{2}(5 - 3x)^{-1/2}(-3) = \frac{-3}{2\sqrt{5 - 3x}}$

16. $g(x) = \sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = |x - 1|$

$$g'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \end{cases}$$

18. $f(x) = -3(2 - 9x)^{1/4}$

$f'(x) = -\frac{3}{4}(2 - 9x)^{-3/4}(-9) = \frac{27}{4(2 - 9x)^{3/4}}$

20. $s(t) = (t^2 + 3t - 1)^{-1}$

$$s'(t) = -1(t^2 + 3t - 1)^{-2}(2t + 3) \\ = \frac{-(2t + 3)}{(t^2 + 3t - 1)^2}$$

22. $y = -5(t + 3)^{-3}$

$y' = 15(t + 3)^{-4} = \frac{15}{(t + 3)^4}$

24. $g(t) = (t^2 - 2)^{-1/2}$

$g'(t) = -\frac{1}{2}(t^2 - 2)^{-3/2}(2t) = -\frac{t}{(t^2 - 2)^{3/2}}$

26. $f(x) = x(3x - 9)^3$

$$f'(x) = x[3(3x - 9)^2(3)] + (3x - 9)^3(1) \\ = (3x - 9)^2[9x + 3x - 9] \\ = 27(x - 3)^2(4x - 3)$$

28. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

$$y' = \frac{1}{2}x^2\left(\frac{1}{2}(16 - x^2)^{-1/2}(-2x)\right) + x(16 - x^2)^{1/2} \\ = \frac{-x^3}{2\sqrt{16 - x^2}} + x\sqrt{16 - x^2} \\ = \frac{-x(3x^2 - 32)}{2\sqrt{16 - x^2}}$$

30. $y = \frac{x}{\sqrt{x^4 + 4}}$

$$y' = \frac{(x^4 + 4)^{1/2}(1) - x\frac{1}{2}(x^4 + 4)^{-1/2}(4x^3)}{x^4 + 4} \\ = \frac{x^4 + 4 - 2x^4}{(x^4 + 4)^{3/2}} = \frac{4 - x^4}{(x^4 + 4)^{3/2}}$$

32. $h(t) = \left(\frac{t^2}{t^3 + 2}\right)^2$

$$h'(t) = 2\left(\frac{t^2}{t^3 + 2}\right)\left(\frac{(t^3 + 2)(2t) - t^2(3t^2)}{(t^3 + 2)^2}\right) \\ = \frac{2t^2(4t - t^4)}{(t^3 + 2)^3} = \frac{2t^3(4 - t^3)}{(t^3 + 2)^3}$$

$$34. g(x) = \left(\frac{3x^2 - 2}{2x + 3}\right)^3$$

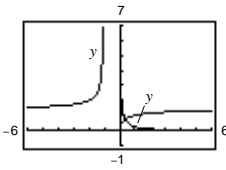
$$g'(x) = 3\left(\frac{3x^2 - 2}{2x + 3}\right)^2 \left(\frac{(2x + 3)(6x) - (3x^2 - 2)(2)}{(2x + 3)^2}\right)$$

$$= \frac{3(3x^2 - 2)^2(6x^2 + 18x + 4)}{(2x + 3)^4} = \frac{6(3x^2 - 2)^2(3x^2 + 9x + 2)}{(2x + 3)^4}$$

$$36. y = \sqrt{\frac{2x}{x+1}}$$

$$y' = \frac{1}{\sqrt{2x}(x+1)^{3/2}}$$

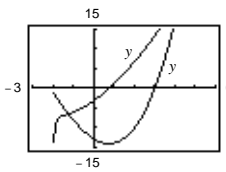
y' has no zeros.



$$40. y = (t^2 - 9)\sqrt{t+2}$$

$$y' = \frac{5t^2 + 8t - 9}{2\sqrt{t+2}}$$

The zero of y' corresponds to the point on the graph of y where the tangent line is horizontal.



$$44. y = x^2 \tan \frac{1}{x}$$

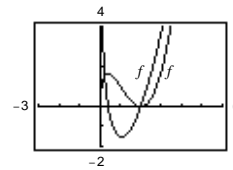
$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.

$$38. f(x) = \sqrt{x}(2-x)^2$$

$$f'(x) = \frac{(x-2)(5x-2)}{2\sqrt{x}}$$

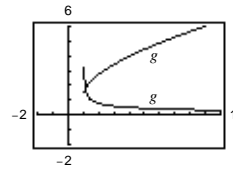
The zeros of f' correspond to the points on the graph of f where the tangent lines are horizontal.



$$42. g(x) = \sqrt{x-1} + \sqrt{x+1}$$

$$g'(x) = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$$

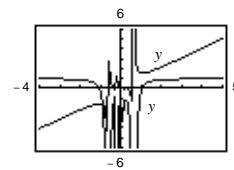
g' has no zeros.



$$44. y = x^2 \tan \frac{1}{x}$$

$$\frac{dy}{dx} = 2x \tan \frac{1}{x} - \sec^2 \frac{1}{x}$$

The zeros of y' correspond to the points on the graph of y where the tangent lines are horizontal.



$$46. (a) \quad y = \sin 3x$$

$$y' = 3 \cos 3x$$

$$y'(0) = 3$$

$$3 \text{ cycles in } [0, 2\pi]$$

$$(b) \quad y = \sin\left(\frac{x}{2}\right)$$

$$y' = \left(\frac{1}{2}\right) \cos\left(\frac{x}{2}\right)$$

$$y'(0) = \frac{1}{2}$$

Half cycle in $[0, 2\pi]$

The slope of $\sin ax$ at the origin is a .

48. $y = \sin \pi x$

$$\frac{dy}{dx} = \pi \cos \pi x$$

52. $y = \cos(1 - 2x)^2 = \cos((1 - 2x)^2)$

$$y' = -\sin(1 - 2x)^2(2(1 - 2x)(-2)) = 4(1 - 2x) \sin(1 - 2x)^2$$

54. $g(\theta) = \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right)$

$$\begin{aligned} g'(\theta) &= \sec\left(\frac{1}{2}\theta\right) \sec^2\left(\frac{1}{2}\theta\right) \frac{1}{2} + \tan\left(\frac{1}{2}\theta\right) \sec\left(\frac{1}{2}\theta\right) \tan\left(\frac{1}{2}\theta\right) \frac{1}{2} \\ &= \frac{1}{2} \sec\left(\frac{1}{2}\theta\right) \left[\sec^2\left(\frac{1}{2}\theta\right) + \tan^2\left(\frac{1}{2}\theta\right) \right] \end{aligned}$$

56. $g(v) = \frac{\cos v}{\csc v} = \cos v \cdot \sin v$

$$g'(v) = \cos v(\cos v) + \sin v(-\sin v) = \cos^2 v - \sin^2 v = \cos 2v$$

58. $y = 2 \tan^3 x$

$$y' = 6 \tan^2 x \cdot \sec^2 x$$

50. $h(x) = \sec(x^2)$

$$h'(x) = 2x \sec(x^2) \tan(x^2)$$

60. $g(t) = 5 \cos^2 \pi t = 5(\cos \pi t)^2$

$$\begin{aligned} g'(t) &= 10 \cos \pi t (-\sin \pi t)(\pi) \\ &= -10\pi(\sin \pi t)(\cos \pi t) = -5\pi \sin 2\pi t \end{aligned}$$

62. $h(t) = 2 \cot^2(\pi t + 2)$

$$\begin{aligned} h'(t) &= 4 \cot(\pi t + 2)(-\csc^2(\pi t + 2)(\pi)) \\ &= -4\pi \cot(\pi t + 2) \csc^2(\pi t + 2) \end{aligned}$$

64. $y = 3x - 5 \cos(\pi x)^2$

$$= 3x - 5 \cos(\pi^2 x^2)$$

$$\begin{aligned} \frac{dy}{dx} &= 3 + 5 \sin(\pi^2 x^2)(2\pi^2 x) \\ &= 3 + 10\pi^2 x \sin(\pi x)^2 \end{aligned}$$

66. $y = \sin x^{1/3} + (\sin x)^{1/3}$

$$\begin{aligned} y' &= \cos x^{1/3} \left(\frac{1}{3} x^{-2/3} \right) + \frac{1}{3} (\sin x)^{-2/3} \cos x \\ &= \frac{1}{3} \left[\frac{\cos x^{1/3}}{x^{2/3}} + \frac{\cos x}{(\sin x)^{2/3}} \right] \end{aligned}$$

68. $y = (3x^3 + 4x)^{1/5}, (2, 2)$

$$\begin{aligned} y' &= \frac{1}{5} (3x^3 + 4x)^{-4/5} (9x^2 + 4) \\ &= \frac{9x^2 + 4}{5(3x^3 + 4x)^{4/5}} \end{aligned}$$

$$y'(2) = \frac{1}{2}$$

70. $f(x) = \frac{1}{(x^2 - 3x)^2} = (x^2 - 3x)^{-2}, \left(4, \frac{1}{16}\right)$

$$f'(x) = -2(x^2 - 3x)^{-3}(2x - 3) = \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$

$$f'(4) = -\frac{5}{32}$$

72. $f(x) = \frac{x + 1}{2x - 3}, (2, 3)$

$$f'(x) = \frac{(2x - 3)(1) - (x + 1)(2)}{(2x - 3)^2} = \frac{-5}{(2x - 3)^2}$$

$$f'(2) = -5$$

74. $y = \frac{1}{x} + \sqrt{\cos x}, \left(\frac{\pi}{2}, \frac{2}{\pi}\right)$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$y'(\pi/2)$ is undefined.

76. (a) $f(x) = \frac{1}{3}x\sqrt{x^2 + 5}$, $(2, 2)$

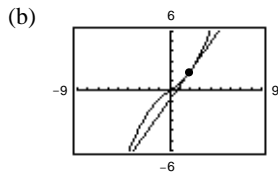
$$f'(x) = \frac{1}{3}x \left[\frac{1}{2}(x^2 + 5)^{-1/2}(2x) \right] + \frac{1}{3}(x^2 + 5)^{1/2}$$

$$= \frac{x^2}{3\sqrt{x^2 + 5}} + \frac{1}{3}\sqrt{x^2 + 5}$$

$$f'(2) = \frac{4}{3(3)} + \frac{1}{3}(3) = \frac{13}{9}$$

Tangent line:

$$y - 2 = \frac{13}{9}(x - 2) \quad 13x - 9y - 8 = 0$$



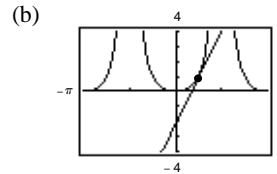
78. (a) $f(x) = \tan^2 x$, $\left(\frac{\pi}{4}, 1\right)$

$$f'(x) = 2 \tan x \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = 2(1)(2) = 4$$

Tangent line:

$$y - 1 = 4\left(x - \frac{\pi}{4}\right) \quad 4x - y + (1 - \pi) = 0$$



80. $f(x) = (x - 2)^{-1}$

$$f'(x) = -(x - 2)^{-2} = \frac{-1}{(x - 2)^2}$$

$$f''(x) = 2(x - 2)^{-3} = \frac{2}{(x - 2)^3}$$

82. $f(x) = \sec^2 \pi x$

$$f'(x) = 2 \sec \pi x (\pi \sec \pi x \tan \pi x)$$

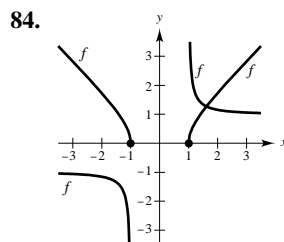
$$= 2\pi \sec^2 \pi x \tan \pi x$$

$$f''(x) = 2\pi \sec^2 \pi x (\sec^2 \pi x)(\pi) + 2\pi \tan \pi x (2\pi \sec^2 \pi x \tan \pi x)$$

$$= 2\pi^2 \sec^4 \pi x + 4\pi^2 \sec^2 \pi x \tan^2 \pi x$$

$$= 2\pi^2 \sec^2 \pi x (\sec^2 \pi x + 2 \tan^2 \pi x)$$

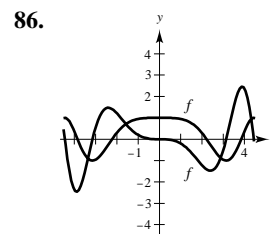
$$= 2\pi^2 \sec^2 \pi x (3 \sec^2 \pi x - 2)$$



f is decreasing on $(-\infty, -1)$ so f' must be negative there. f is increasing on $(1, \infty)$ so f' must be positive there.

88. $g(x) = f(x^2)$

$$g'(x) = f'(x^2)(2x) \quad g'(x) = 2x f'(x^2)$$



The zeros of f' correspond to the points where the graph of f has horizontal tangents.

90. (a) $g(x) = \sin^2 x + \cos^2 x = 1 \quad g'(x) = 0$

$$g'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$$

(b) $\tan^2 x + 1 = \sec^2 x$

$$g(x) + 1 = f(x)$$

Taking derivatives of both sides,

$$g'(x) = f'(x).$$

Equivalently, $f'(x) = 2 \sec x \cdot \sec x \cdot \tan x$ and

$g'(x) = 2 \tan x \cdot \sec^2 x$, which are the same.

94. $y = A \cos \omega t$

(a) Amplitude: $A = \frac{3.5}{2} = 1.75$

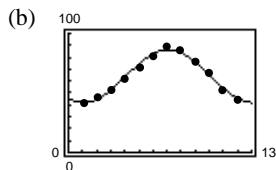
$$y = 1.75 \cos \omega t$$

Period: 10 $\omega = \frac{2\pi}{10} = \frac{\pi}{5}$

$$y = 1.75 \cos \frac{\pi t}{5}$$

96. (a) Using a graphing utility, or by trial and error, you obtain a model of the form

$$T(t) = 64.18 - 22.15 \sin\left(\frac{\pi t}{6} + 1\right)$$



92. $y = \frac{1}{3} \cos 12t - \frac{1}{4} \sin 12t$

$$v = y' = \frac{1}{3}[-12 \sin 12t] - \frac{1}{4}[12 \cos 12t]$$

$$= -4 \sin 12t - 3 \cos 12t$$

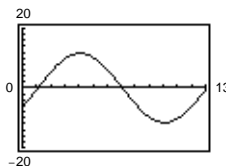
When $t = \pi/8$, $y = 0.25$ feet and $v = 4$ feet per second.

(b) $v = y' = 1.75 \left[-\frac{\pi}{5} \sin \frac{\pi t}{5} \right]$

$$= -0.35 \pi \sin \frac{\pi t}{5}$$

(c) $T'(t) = -22.15 \cos\left(\frac{\pi t}{6} + 1\right) \left(\frac{\pi}{6}\right)$

$$= -11.60 \cos\left(\frac{\pi t}{6} + 1\right)$$



(d) The temperature changes most rapidly when $t \approx 4.1$ (April) and $t \approx 10.1$ (October). The temperature changes most slowly ($T'(t) = 0$) when $t \approx 1.1$ (January) and $t \approx 7.1$ (July).

98. (a) $g(x) = f(x) - 2 \quad g'(x) = f'(x)$

(b) $h(x) = 2f(x) \quad h'(x) = 2f'(x)$

(c) $r(x) = f(-3x) \quad r'(x) = f'(-3x)(-3) = -3f'(-3x)$

Hence, you need to know $f'(-3x)$.

$$r'(0) = -3f'(0) = (-3)\left(-\frac{1}{3}\right) = 1$$

$$r'(-1) = -3f'(3) = (-3)(-4) = 12$$

(d) $s(x) = f(x + 2) \quad s'(x) = f'(x + 2)$

Hence, you need to know $f'(x + 2)$.

$$s'(-2) = f'(0) = -\frac{1}{3}, \text{ etc.}$$

| | | | | | | |
|---------|----------------|---------------|----------------|----|----|----|
| x | -2 | -1 | 0 | 1 | 2 | 3 |
| $f'(x)$ | 4 | $\frac{2}{3}$ | $-\frac{1}{3}$ | -1 | -2 | -4 |
| $g'(x)$ | 4 | $\frac{2}{3}$ | $-\frac{1}{3}$ | -1 | -2 | -4 |
| $h'(x)$ | 8 | $\frac{4}{3}$ | $-\frac{2}{3}$ | -2 | -4 | -8 |
| $r'(x)$ | | 12 | 1 | | | |
| $s'(x)$ | $-\frac{1}{3}$ | -1 | -2 | -4 | | |

100. $f(x + p) = f(x)$ for all x .

- (a) Yes, $f'(x + p) = f'(x)$, which shows that f' is periodic as well.
 (b) Yes, let $g(x) = f(2x)$, so $g'(x) = 2f'(2x)$.
 Since f' is periodic, so is g' .

102. If $f(-x) = -f(x)$, then

$$\begin{aligned}\frac{d}{dx}[f(-x)] &= \frac{d}{dx}[-f(x)] \\ f'(-x)(-1) &= -f'(x) \\ f'(-x) &= f'(x).\end{aligned}$$

Thus, $f'(x)$ is even.

104. $|u| = \sqrt{u^2}$

$$\frac{d}{dx}[|u|] = \frac{d}{dx}[\sqrt{u^2}] = \frac{1}{2}(u^2)^{-1/2}(2uu') = \frac{uu'}{\sqrt{u^2}} = u' \frac{u}{|u|}, u \neq 0$$

106. $f(x) = |x^2 - 4|$

$$f'(x) = 2x \left(\frac{x^2 - 4}{|x^2 - 4|} \right), x \neq \pm 2$$

110. (a) $f(x) = \sec(2x)$

$$f'(x) = 2(\sec 2x)(\tan 2x)$$

$$\begin{aligned}f''(x) &= 2[2(\sec 2x)(\tan 2x)] \tan 2x + 2(\sec 2x)(\sec^2 2x)(2) \\ &= 4[(\sec 2x)(\tan^2 2x) + \sec^3 2x]\end{aligned}$$

$$f\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{3}\right) = 2$$

$$f'\left(\frac{\pi}{6}\right) = 2 \sec\left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3}\right) = 4\sqrt{3}$$

$$f''\left(\frac{\pi}{6}\right) = 4[2(3) + 2^3] = 56$$

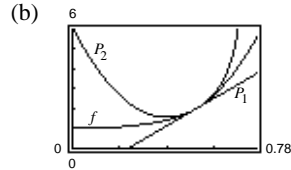
$$P_1(x) = 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2$$

$$\begin{aligned}P_2(x) &= \frac{1}{2}(56)\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2 \\ &= 28\left(x - \frac{\pi}{6}\right)^2 + 4\sqrt{3}\left(x - \frac{\pi}{6}\right) + 2\end{aligned}$$

(c) P_2 is a better approximation than P_1 .

108. $f(x) = |\sin x|$

$$f'(x) = \cos x \left(\frac{\sin x}{|\sin x|} \right), x \neq k\pi$$



(d) The accuracy worsens as you move away from $x = \pi/6$.

112. False. If $f(x) = \sin^2 2x$, then $f'(x) = 2(\sin 2x)(2 \cos 2x)$.

114. False. First apply the Product Rule.

Section 2.5 Implicit Differentiations

2. $x^2 - y^2 = 16$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

6. $x^2y + y^2x = -2$

$$x^2y' + 2xy + y^2 + 2yxy' = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

10. $2 \sin x \cos y = 1$

$$2[\sin x(-\sin y)y' + \cos y(\cos x)] = 0$$

$$y' = \frac{\cos x \cos y}{\sin x \sin y} \\ = \cot x \cot y$$

14. $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$

$$y' = \frac{1}{1 - \csc^2 y}$$

$$= \frac{1}{-\cot^2 y} = -\tan^2 y$$

18. (a) $(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$

$$(x - 2)^2 + (y + 3)^2 = 4 \text{ (Circle)}$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(c) Explicitly:

$$\frac{dy}{dx} = \pm \frac{1}{2} [4 - (x - 2)^2]^{-1/2} (-2)(x - 2)$$

$$= \frac{\mp(x - 2)}{(\sqrt{4 - (x - 2)^2})^2}$$

$$= \frac{-(x - 2)}{\pm \sqrt{4 - (x - 2)^2}}$$

$$= \frac{-(x - 2)}{-3 \pm \sqrt{4 - (x - 2)^2} + 3}$$

$$= \frac{-(x - 2)}{y + 3}$$

4. $x^3 + y^3 = 8$

$$3x^2 + 3y^2y' = 0$$

$$y' = -\frac{x^2}{y^2}$$

8. $(xy)^{1/2} - x + 2y = 0$

$$\frac{1}{2}(xy)^{-1/2}(xy' + y) - 1 + 2y' = 0$$

$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} - 1 + 2y' = 0$$

$$xy' + y - 2\sqrt{xy} + 4\sqrt{xy}y' = 0$$

$$y' = \frac{2\sqrt{xy} - y}{4\sqrt{xy} + x}$$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$

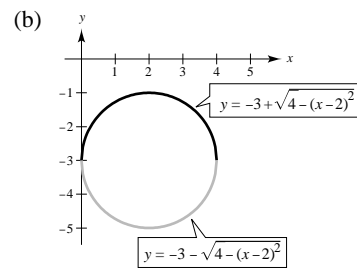
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$

$$y' = \frac{\cos \pi x}{\sin \pi y}$$

16. $x = \sec \frac{1}{y}$

$$1 = -\frac{y'}{y^2} \sec \frac{1}{y} \tan \frac{1}{y}$$

$$y' = \frac{-y^2}{\sec(1/y) \tan(1/y)} = -y^2 \cos\left(\frac{1}{y}\right) \cot\left(\frac{1}{y}\right)$$



(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

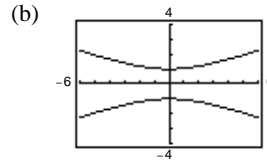
$$(2y + 6)y' = -2(x - 2)$$

$$y' = \frac{-(x - 2)}{y + 3}$$

20. (a) $9y^2 = x^2 + 9$

$$y^2 = \frac{x^2}{9} + 1 = \frac{x^2 + 9}{9}$$

$$y = \frac{\pm\sqrt{x^2 + 9}}{3}$$



(c) Explicitly: $\frac{dy}{dx} = \frac{\pm\frac{1}{2}(x^2 + 9)^{-1/2}(2x)}{3} = \frac{\pm x}{3\sqrt{x^2 + 9}} = \frac{\pm x}{3(\pm 3y)} = \frac{x}{9y}$

(d) Implicitly: $9y^2 - x^2 = 9$

$$18yy' - 2x = 0$$

$$18yy' = 2x$$

$$y' = \frac{2x}{18y} = \frac{x}{9y}$$

22. $x^2 - y^3 = 0$

$$2x - 3y^2y' = 0$$

$$y' = \frac{2x}{3y^2}$$

At (1, 1): $y' = \frac{2}{3}$.

24. $(x + y)^3 = x^3 + y^3$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At (-1, 1): $y' = -1$.

26. $x^3 + y^3 = 4xy + 1$

$$3x^2 + 3y^2y' = 4xy' + 4y$$

$$(3y^2 - 4x)y' = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{(3y^2 - 4x)}$$

At (2, 1), $y' = \frac{4 - 12}{3 - 8} = \frac{8}{5}$

28. $x \cos y = 1$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x} \cot y = \frac{\cot y}{x}$$

At $(2, \frac{\pi}{3})$: $y' = \frac{1}{2\sqrt{3}}$.

30. $(4 - x)y^2 = x^3$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At (2, 2): $y' = 2$.

32. $x^3 + y^3 - 6xy = 0$

$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $(\frac{4}{3}, \frac{8}{3})$: $y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$.

34. $\cos y = x$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{-1}{\sin y}, 0 < y < \pi$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}}, -1 < x < 1$$

36. $x^2y^2 - 2x = 3$

$$2x^2yy' + 2xy^2 - 2 = 0$$

$$x^2yy' + xy^2 - 1 = 0$$

$$y' = \frac{1 - xy^2}{x^2y}$$

$$2xyy' + x^2(y')^2 + x^2yy'' + 2xyy' + y^2 = 0$$

$$4xyy' + x^2(y')^2 + x^2yy'' + y^2 = 0$$

$$\frac{4 - 4xy^2}{x} + \frac{(1 - xy^2)^2}{x^2y^2} + x^2yy'' + y^2 = 0$$

$$4xy^2 - 4x^2y^4 + 1 - 2xy^2 + x^2y^4 + x^4y^3y'' + x^2y^4 = 0$$

$$x^4y^3y'' = 2x^2y^4 - 2xy^2 - 1$$

$$y'' = \frac{2x^2y^4 - 2xy^2 - 1}{x^4y^3}$$

38. $1 - xy = x - y$

$$y - xy = x - 1$$

$$y = \frac{x - 1}{1 - x} = -1$$

$$y' = 0$$

$$y'' = 0$$

40. $y^2 = 4x$

$$2yy' = 4$$

$$y' = \frac{2}{y}$$

$$y'' = -2y^{-2}y' = \left[\frac{-2}{y^2} \right] \cdot \frac{2}{y} = \frac{-4}{y^3}$$

42. $y^2 = \frac{x - 1}{x^2 + 1}$

$$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$$

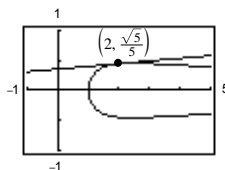
$$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$$

$$\text{At } \left(2, \frac{\sqrt{5}}{5} \right): y' = \frac{1 + 4 - 4}{\left[\frac{(2\sqrt{5})}{5} \right] (4 + 1)^2} = \frac{1}{10\sqrt{5}}$$

$$\text{Tangent line: } y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$$

$$10\sqrt{5}y - 10 = x - 2$$

$$x - 10\sqrt{5}y + 8 = 0$$



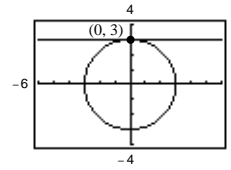
44. $x^2 + y^2 = 9$

$$y' = \frac{-x}{y}$$

 At $(0, 3)$:

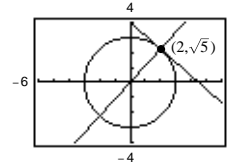
Tangent line: $y = 3$

Normal line: $x = 0$.


 At $(2, \sqrt{5})$:

Tangent line: $y - \sqrt{5} = \frac{-2}{\sqrt{5}}(x - 2)$ $2x + \sqrt{5}y - 9 = 0$

Normal line: $y - \sqrt{5} = \frac{\sqrt{5}}{2}(x - 2)$ $\sqrt{5}x - 2y = 0$.



46. $y^2 = 4x$

$2yy' = 4$

$y' = \frac{2}{y} = 1$ at $(1, 2)$

Equation of normal at $(1, 2)$ is $y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal and at a distance of 4 units from $(1, 2)$. Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$

$$2(x - 1)^2 = 16$$

$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

Equations: $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$

48. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2yy' - 8 + 4y' = 0$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$

Horizontal tangents occur when $x = 1$:

$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$

$y^2 + 4y = y(y + 4) = 0 \quad y = 0, -4$

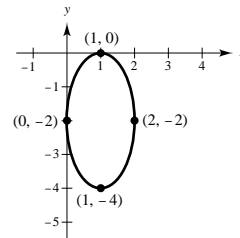
Horizontal tangents: $(1, 0), (1, -4)$.

Vertical tangents occur when $y = -2$:

$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$

$4x^2 - 8x = 4x(x - 2) = 0 \quad x = 0, 2$

Vertical tangents: $(0, -2), (2, -2)$.



50. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$$2x^2 + 3x^3 = 5 \quad \text{and} \quad 3x^3 + 2x^2 - 5 = 0$$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$

$$\begin{array}{ll} \underline{y^2 = x^3:} & \underline{2x^2 + 3y^2 = 5:} \\ 2yy' = 3x^2 & 4x + 6yy' = 0 \\ y' = \frac{3x^2}{2y} & y' = -\frac{2x}{3y} \end{array}$$

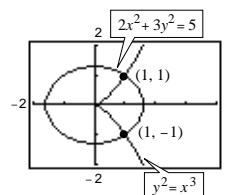
At $(1, 1)$, the slopes are:

$$y' = \frac{3}{2} \qquad y' = -\frac{2}{3}$$

At $(1, -1)$, the slopes are:

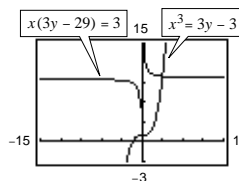
$$y' = -\frac{3}{2} \qquad y' = \frac{2}{3}$$

Tangents are perpendicular.



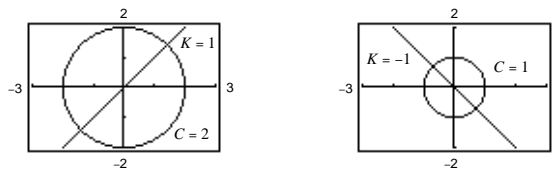
52. Rewriting each equation and differentiating,

$$\begin{array}{ll} x^3 = 3(y - 1) & x(3y - 29) = 3 \\ y = \frac{x^3}{3} + 1 & y = \frac{1}{3}\left(\frac{3}{x} + 29\right) \\ y' = x^2 & y' = -\frac{1}{x^2} \end{array}$$



For each value of x , the derivatives are negative reciprocals of each other. Thus, the tangent lines are orthogonal at both points of intersection.

54. $x^2 + y^2 = C^2 \quad y = Kx$
 $2x + 2yy' = 0 \quad y' = K$
 $y' = -\frac{x}{y}$



At the point of intersection (x, y) the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

56. $x^2 - 3xy^2 + y^3 = 10$

(a) $2x - 3y^2 - 6xyy' + 3y^2y' = 0$
 $(-6xy + 3y^2)y' = 3y^2 - 2x$
 $y' = \frac{3y^2 - 2x}{3y^2 - 6xy}$

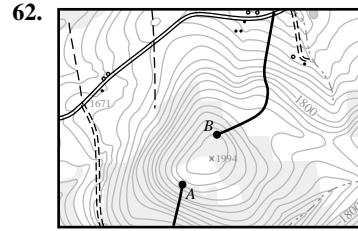
(b) $2x \frac{dx}{dt} - 3y^2 \frac{dx}{dt} - 6xy \frac{dy}{dt} + 3y^2 \frac{dy}{dt} = 0$
 $(2x - 3y^2) \frac{dx}{dt} = (6xy - 3y^2) \frac{dy}{dt}$

58. (a) $4 \sin x \cos y = 1$

$4 \sin x(-\sin y)y' + 4 \cos x \cos y = 0$
 $y' = \frac{\cos x \cos y}{\sin x \sin y}$

(b) $4 \sin x(-\sin y) \frac{dy}{dt} + 4 \cos x \frac{dx}{dt} \cos y = 0$
 $\cos x \cos y \frac{dx}{dt} = \sin x \sin y \frac{dy}{dt}$

60. Given an implicit equation, first differentiate both sides with respect to x . Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y' .



Use starting point B.

64. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0) :

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$$x\text{-intercept: } (x_0 + \sqrt{x_0}\sqrt{y_0}, 0)$$

$$y\text{-intercept: } (0, y_0 + \sqrt{x_0}\sqrt{y_0})$$

Sum of intercepts:

$$(x_0 + \sqrt{x_0}\sqrt{y_0}) + (y_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c.$$

Section 2.6 Related Rates

2. $y = 2(x^2 - 3x)$

$$\frac{dy}{dt} = (4x - 6) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4x - 6} \frac{dy}{dt}$$

(a) When $x = 3$ and $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = [4(3) - 6](2) = 12$

(b) When $x = 1$ and $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = \frac{1}{4(1) - 6}(5) = -\frac{5}{2}$

4. $x^2 + y^2 = 25$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \left(-\frac{x}{y}\right) \frac{dx}{dt}$$

$$\frac{dx}{dt} = \left(-\frac{y}{x}\right) \frac{dy}{dt}$$

(a) When $x = 3$, $y = 4$, and $dx/dt = 8$,

$$\frac{dy}{dt} = -\frac{3}{4}(8) = -6$$

(b) When $x = 4$, $y = 3$, and $dy/dt = -2$,

$$\frac{dx}{dt} = -\frac{3}{4}(-2) = \frac{3}{2}.$$

$$6. \quad y = \frac{1}{1+x^2}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \left[\frac{-2x}{(1+x^2)^2} \right] \frac{dx}{dt}$$

(a) When $x = -2$,

$$\frac{dy}{dt} = \frac{-2(-2)(2)}{25} = \frac{8}{25} \text{ cm/sec.}$$

(b) When $x = 0$,

$$\frac{dy}{dt} = 0 \text{ cm/sec.}$$

(c) When $x = 2$,

$$\frac{dy}{dt} = \frac{-2(2)(2)}{25} = \frac{-8}{25} \text{ cm/sec.}$$

$$10. \quad (a) \quad \frac{dx}{dt} \text{ negative} \quad \frac{dy}{dt} \text{ negative}$$

$$(b) \quad \frac{dy}{dt} \text{ positive} \quad \frac{dx}{dt} \text{ positive}$$

$$14. \quad D = \sqrt{x^2 + y^2} = \sqrt{x^2 + \sin^2 x}$$

$$\frac{dx}{dt} = 2$$

$$\frac{dD}{dt} = \frac{1}{2}(x^2 + \sin^2 x)^{-1/2}(2x + 2 \sin x \cos x) \frac{dx}{dt} = \frac{x + \sin x \cos x}{\sqrt{x^2 + \sin^2 x}} \frac{dx}{dt} = \frac{2 + 2 \sin x \cos x}{\sqrt{x^2 + \sin^2 x}}$$

$$16. \quad A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

If dr/dt is constant, dA/dt is not constant.

$$\frac{dA}{dt} \text{ depends on } r \text{ and } \frac{dr}{dt}.$$

$$20. \quad V = x^3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

(a) When $x = 1$,

$$\frac{dV}{dt} = 3(1)^2(3) = 9 \text{ cm}^3/\text{sec.}$$

$$8. \quad y = \sin x$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = \cos x \frac{dx}{dt}$$

(a) When $x = \pi/6$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{6} \right) (2) = \sqrt{3} \text{ cm/sec.}$$

(b) When $x = \pi/4$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{4} \right) (2) = \sqrt{2} \text{ cm/sec.}$$

(c) When $x = \pi/3$,

$$\frac{dy}{dt} = \left(\cos \frac{\pi}{3} \right) (2) = 1 \text{ cm/sec.}$$

12. Answers will vary. See page 145.

$$18. \quad V = \frac{4}{3}\pi r^3$$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$(a) \quad \text{When } r = 6, \frac{dV}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{min.}$$

$$\text{When } r = 24, \frac{dV}{dt} = 4\pi(24)^2(2) = 4608\pi \text{ in}^3/\text{min.}$$

(b) If dr/dt is constant, dV/dt is proportional to r^2 .

(b) When $x = 10$,

$$\frac{dV}{dt} = 3(10)^2(3) = 900 \text{ cm}^3/\text{sec.}$$

$$22. V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$$

$$\frac{dr}{dt} = 2$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

(a) When $r = 6$,

$$\frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi \text{ in}^3/\text{min.}$$

(b) When $r = 24$,

$$\frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi \text{ in}^3/\text{min.}$$

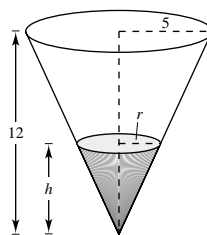
$$24. V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \frac{25}{144} h^3 = \frac{25\pi}{3(144)} h^3$$

(By similar triangles, $\frac{r}{5} = \frac{h}{12}$ $r = \frac{5}{12}h$.)

$$\frac{dV}{dt} = 10$$

$$\frac{dV}{dt} = \frac{25\pi}{144} h^2 \frac{dh}{dt} \quad \frac{dh}{dt} = \left(\frac{144}{25\pi h^2}\right) \frac{dV}{dt}$$

$$\text{When } h = 8, \frac{dh}{dt} = \frac{144}{25\pi(64)}(10) = \frac{9}{10\pi} \text{ ft/min.}$$

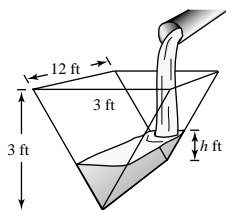


$$26. V = \frac{1}{2}bh(12) = 6bh = 6h^2 \text{ (since } b = h\text{)}$$

$$(a) \frac{dV}{dt} = 12h \frac{dh}{dt} \quad \frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt}$$

$$\text{When } h = 1 \text{ and } \frac{dV}{dt} = 2, \frac{dh}{dt} = \frac{1}{12(1)}(2) = \frac{1}{6} \text{ ft/min}$$

$$(b) \text{ If } \frac{dh}{dt} = \frac{3}{8} \text{ and } h = 2, \text{ then } \frac{dV}{dt} = 12(2)\left(\frac{3}{8}\right) = 9 \text{ ft}^3/\text{min.}$$



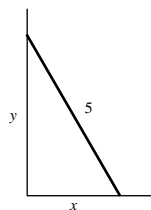
$$28. x^2 + y^2 = 25$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \cdot \frac{dy}{dt} = -\frac{0.15y}{x} \text{ since } \frac{dy}{dt} = 0.15.$$

When $x = 2.5$,

$$y = \sqrt{18.75}, \frac{dx}{dt} = -\frac{\sqrt{18.75}}{2.5} 0.15 \approx -0.26 \text{ m/sec}$$



30. Let L be the length of the rope.

$$(a) L^2 = 144 + x^2$$

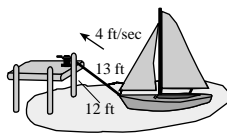
$$2L \frac{dL}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{L}{x} \cdot \frac{dL}{dt} = -\frac{4L}{x} \text{ since } \frac{dL}{dt} = -4 \text{ ft/sec.}$$

When $L = 13$,

$$x = \sqrt{L^2 - 144} = \sqrt{169 - 144} = 5$$

$$\frac{dx}{dt} = -\frac{4(13)}{5} = -\frac{52}{5} = -10.4 \text{ ft/sec.}$$



$$(b) \text{ If } \frac{dx}{dt} = -4, \text{ and } L = 13,$$

$$\frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt}$$

$$= \frac{5}{13}(-4)$$

$$= -\frac{20}{13} \text{ ft/sec}$$

As L \rightarrow 0, $\frac{dL}{dt}$ increases.

Speed of the boat increases as it approaches the dock.

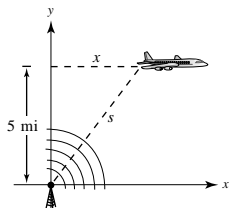
32. $x^2 + y^2 = s^2$

$$2x \frac{dx}{dt} + 0 = 2s \frac{ds}{dt} \quad \left(\text{since } \frac{dy}{dt} = 0 \right)$$

$$\frac{dx}{dt} = \frac{s}{x} \frac{ds}{dt}$$

When $s = 10$, $x = \sqrt{100 - 25} = \sqrt{75} = 5\sqrt{3}$

$$\frac{dx}{dt} = \frac{10}{5\sqrt{3}}(-240) = \frac{-480}{\sqrt{3}} = -160\sqrt{3} \approx -277.13 \text{ mph.}$$



36. (a) $\frac{20}{6} = \frac{y}{y-x}$

$$20y - 20x = 6y$$

$$14y = 20x$$

$$y = \frac{10}{7}x$$

$$\frac{dx}{dt} = -5$$

$$\frac{dy}{dt} = \frac{10}{7} \frac{dx}{dt} = \frac{10}{7}(-5) = \frac{-50}{7} \text{ ft/sec}$$

(b) $\frac{d(y-x)}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = \frac{-50}{7} - (-5) = \frac{-50}{7} + \frac{35}{7} = \frac{-15}{7} \text{ ft/sec}$

38. $x(t) = \frac{3}{5} \sin \pi t$, $x^2 + y^2 = 1$

(a) Period: $\frac{2\pi}{\pi} = 2$ seconds

(b) When $x = \frac{3}{5}$, $y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ m.

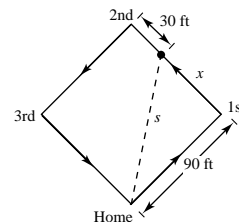
Lowest point: $\left(0, \frac{4}{5}\right)$

34. $s^2 = 90^2 + x^2$

$$x = 60$$

$$\frac{dx}{dt} = 28$$

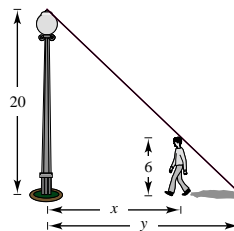
$$\frac{ds}{dt} = \frac{x}{s} \cdot \frac{dx}{dt}$$



When $x = 60$,

$$s = \sqrt{90^2 + 60^2} = 30\sqrt{13}$$

$$\frac{ds}{dt} = \frac{60}{30\sqrt{13}}(28) = \frac{56}{\sqrt{13}} \approx 15.53 \text{ ft/sec.}$$



(c) When $x = \frac{3}{10}$, $y = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ and

$$\frac{3}{10} = \frac{3}{5} \sin \pi t \quad \sin \pi t = \frac{1}{2} \quad t = \frac{1}{6}$$

$$\frac{dx}{dt} = \frac{3}{5} \pi \cos \pi t$$

$$x^2 + y^2 = 1$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$

Thus, $\frac{dy}{dt} = \frac{-3/10}{\sqrt{15}/4} \cdot \frac{3}{5} \pi \cos\left(\frac{\pi}{6}\right)$

$$= \frac{-9\pi}{25\sqrt{5}} = \frac{-9\sqrt{5}\pi}{125}$$

Speed = $\left| \frac{-9\sqrt{5}\pi}{125} \right| \approx 0.5058 \text{ m/sec}$

$$40. \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\frac{dR_1}{dt} = 1$$

$$\frac{dR_2}{dt} = 1.5$$

$$\frac{1}{R^2} \cdot \frac{dR}{dt} = \frac{1}{R_1^2} \cdot \frac{dR_1}{dt} + \frac{1}{R_2^2} \cdot \frac{dR_2}{dt}$$

When $R_1 = 50$ and $R_2 = 75$,

$$R = 30$$

$$\frac{dR}{dt} = (30)^2 \left[\frac{1}{(50)^2} (1) + \frac{1}{(75)^2} (1.5) \right]$$

$$= 0.6 \text{ ohms/sec.}$$

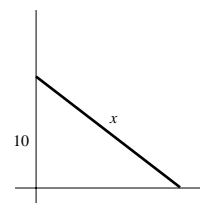
$$44. \quad \sin \theta = \frac{10}{x}$$

$$\frac{dx}{dt} = (-1) \text{ ft/sec}$$

$$\cos \theta \left(\frac{d\theta}{dt} \right) = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-10}{x^2} \frac{dx}{dt} (\sec \theta)$$

$$= \frac{-10}{25^2} (-1) \frac{25}{\sqrt{25^2 - 10^2}} = \frac{10}{25} \frac{1}{5\sqrt{21}} = \frac{2}{25\sqrt{21}} = \frac{2\sqrt{21}}{525} \approx 0.017 \text{ rad/sec}$$



$$46. \quad \tan \theta = \frac{x}{50}$$

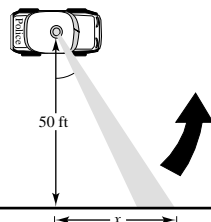
$$\frac{d\theta}{dt} = 30(2\pi) = 60\pi \text{ rad/min} = \pi \text{ rad/sec}$$

$$\sec^2 \theta \left(\frac{d\theta}{dt} \right) = \frac{1}{50} \left(\frac{dx}{dt} \right)$$

$$\frac{dx}{dt} = 50 \sec^2 \theta \left(\frac{d\theta}{dt} \right)$$

(a) When $\theta = 30^\circ$, $\frac{dx}{dt} = \frac{200\pi}{3}$ ft/sec.

(c) When $\theta = 70^\circ$, $\frac{dx}{dt} \approx 427.43\pi$ ft/sec.

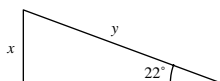


(b) When $\theta = 60^\circ$, $\frac{dx}{dt} = 200\pi$ ft/sec.

$$48. \quad \sin 22^\circ = \frac{x}{y}$$

$$0 = -\frac{x}{y^2} \cdot \frac{dy}{dt} + \frac{1}{y} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{y} \cdot \frac{dy}{dt} = (\sin 22^\circ)(240) \approx 89.9056 \text{ mi/hr}$$



$$42. \quad rg \tan \theta = v^2$$

$32r \tan \theta = v^2$, r is a constant.

$$32r \sec^2 \theta \frac{d\theta}{dt} = 2v \frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{16r}{v} \sec^2 \theta \frac{d\theta}{dt}$$

Likewise, $\frac{d\theta}{dt} = \frac{v}{16r} \cos^2 \theta \frac{dv}{dt}$.

50. (a) $dy/dt = 3(dx/dt)$ means that y changes three times as fast as x changes.

(b) y changes slowly when $x \approx 0$ or $x \approx L$. y changes more rapidly when x is near the middle of the interval.

52. $L^2 = 144 + x^2$; acceleration of the boat $= \frac{d^2x}{dt^2}$.

First derivative: $2L \frac{dL}{dt} = 2x \frac{dx}{dt}$

$$L \frac{dL}{dt} = x \frac{dx}{dt}$$

Second derivative: $L \frac{d^2L}{dt^2} + \frac{dL}{dt} \cdot \frac{dL}{dt} = x \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt}$

$$\frac{d^2x}{dt^2} = \left(\frac{1}{x}\right) \left[L \frac{d^2L}{dt^2} + \left(\frac{dL}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2 \right]$$

When $L = 13$, $x = 5$, $\frac{dx}{dt} = -10.4$, and $\frac{dL}{dt} = -4$ (see Exercise 30). Since $\frac{dL}{dt}$ is constant, $\frac{d^2L}{dt^2} = 0$.

$$\begin{aligned} \frac{d^2x}{dt^2} &= \frac{1}{5} [13(0) + (-4)^2 - (-10.4)^2] \\ &= \frac{1}{5} [16 - 108.16] = \frac{1}{5} [-92.16] = -18.432 \text{ ft/sec}^2 \end{aligned}$$

54. $y(t) = -4.9t^2 + 20$

$$\frac{dy}{dt} = -9.8t$$

$$y(1) = -4.9 + 20 = 15.1$$

$$y'(1) = -9.8$$

By similar triangles, $\frac{20}{x} = \frac{y}{x-12}$

$$20x - 240 = xy.$$

When $y = 15.1$, $20x - 240 = x(15.1)$

$$(20 - 15.1)x = 240$$

$$x = \frac{240}{4.9}.$$

$$20x - 240 = xy$$

$$20 \frac{dx}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{x}{20 - y} \frac{dy}{dt}$$

At $t = 1$, $\frac{dx}{dt} = \frac{240/4.9}{20 - 15.1} (-9.8) \approx -97.96 \text{ m/sec}$.

