CHAPTER 3
Applications of Differentiation

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CHAPTER 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

Solutions to Even-Numbered Exercises

2. $f(x) = \cos\frac{\pi x}{2}$

$f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

4. $f(x) = -3x\sqrt{x+1}$

$$f'(x) = -3x\left[\frac{1}{2}(x+1)^{-1/2}\right] + \sqrt{x+1}(-3)$$
$$= -\frac{3}{2}(x+1)^{-1/2}[x + 2(x+1)]$$
$$= -\frac{3}{2}(x+1)^{-1/2}(3x+2)$$

$f'\left(-\frac{2}{3}\right) = 0$

6. Using the limit definition of the derivative,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{(4 - |x|) - 4}{x - 0} = -1$$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

8. Critical number: $x = 0$.

$x = 0$: neither

10. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum

12. $g(x) = x^2(x^2 - 4) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

14. $f(x) = \frac{4x}{x^2 + 1}$

$$f'(x) = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4(1 - x^2)}{(x^2+1)^2}$$

Critical numbers: $x = \pm 1$

16. $f(\theta) = 2\sec\theta + \tan\theta, 0 < \theta < 2\pi$

$$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta$$
$$= \sec\theta(2\tan\theta + \sec\theta)$$
$$= \sec\theta\left[2\left(\frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\right]$$
$$= \sec^2\theta(2\sin\theta + 1)$$

On $(0, 2\pi)$, critical numbers: $\theta = \frac{7\pi}{6}, \theta = \frac{11\pi}{6}$

18. $f(x) = \frac{2x + 5}{3}, [0, 5]$

$f'(x) = \frac{2}{3}$ No critical numbers

Left endpoint: $\left(0, \frac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

20. $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2 = 2(x + 1)$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

22. $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

24. $g(x) = \sqrt[3]{x}, [-1, 1]$

$g'(x) = \frac{1}{3x^{2/3}}$

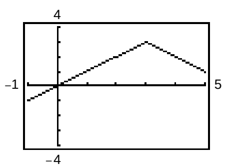
Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

26. $y = 3 - |t - 3|, [-1, 5]$

From the graph, you see that $t = 3$ is a critical number.



Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

28. $h(t) = \frac{t}{t - 2}, [3, 5]$

$h'(t) = \frac{-2}{(t - 2)^2}$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \frac{5}{3}\right)$ Minimum

30. $g(x) = \sec x, \left[-\frac{\pi}{6}, \frac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\frac{\pi}{6}, \frac{2}{\sqrt{3}}\right) \approx \left(-\frac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\frac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

32. $y = x^2 - 2 - \cos x, [-1, 3]$

$y' = 2x - \sin x$

Left endpoint: $(-1, -1.5403)$

Right endpoint: $(3, 7.99)$ Maximum

Critical number: $(0, -3)$ Minimum

34. (a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

36. (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

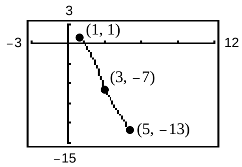
(c) Maximum: $(0, 2)$

(d) Maximum: $\left(1, \sqrt{3}\right)$

38. $f(x) = \begin{cases} 2 - x^2, & 1 \quad x < 3 \\ 2 - 3x, & 3 \quad x \quad 5 \end{cases}$

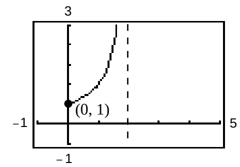
Left endpoint: $(1, 1)$ Maximum

Right endpoint: $(5, -13)$ Minimum

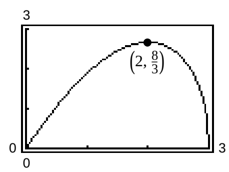


40. $f(x) = \dfrac{2}{2 - x}, [0, 2)$

Left endpoint: $(0, 1)$ Minimum



42. (a)



Maximum: $\left(2, \dfrac{8}{3}\right)$

Minimum: $(0, 0), (3, 0)$

(b) $f(x) = \dfrac{4}{3}x\sqrt{3 - x}, [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right]$$

$$= \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)[-x + 2(3 - x)]$$

$$= \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$f(2) = \dfrac{8}{3}$

Maximum: $\left(2, \dfrac{8}{3}\right)$

44. $f(x) = \dfrac{1}{x^2 + 1}, \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \dfrac{1}{2}$ is the maximum value.

46. $f(x) = \dfrac{1}{x^2 + 1}, [-1, 1]$

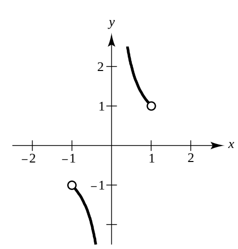
$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4} \text{ (See Exercise 44.)}$$

$$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$$

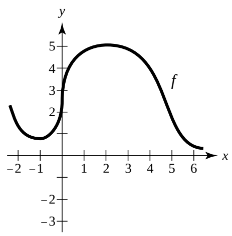
$$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$$

$|f^{(4)}(0)| = 24$ is the maximum value.

48. Let $f(x) = 1/x$. f is continuous on $(0, 1)$ but does not have a maximum. f is also continuous on $(-1, 0)$ but does not have a minimum. This can occur if one of the endpoints is an infinite discontinuity.



50.



52. (a) No

(b) Yes

54. (a) No

(b) Yes

56. $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \quad \theta \quad \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} \text{ (by the Chain Rule)}$$

$$= \frac{v^2 \cos 2\theta}{16}\frac{d\theta}{dt}$$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the spinkler gets the most water.

58. $C = 2x + \dfrac{300{,}000}{x}, 1 \quad x \quad 300$

$$C(1) = 300{,}002$$

$$C(300) = 1600$$

$$C' = 2 - \frac{300{,}000}{x^2} = 0$$

$$2x^2 = 300{,}000$$

$$x^2 = 150{,}000$$

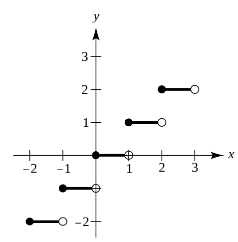
$$x = 100\sqrt{15} \approx 387 > 300 \text{ (outside of interval)}$$

C is minimized when $x = 300$ units.

Yes, if $1 \quad x \quad 400$, then $x = 387$ would minimize C.

60. $f(x) = [\![x]\!]$

The derivative of f is undefined at every integer and is zero at any noninteger real number. All real numbers are critical numbers.



62. True. This is stated in the Extreme Value Theorem.

64. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$$g(x) = f(x - k)$$

$$= (x - k)^2$$

$x = k$ is a critical number of g.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since f is not continuous at $x = 2\pi$.

4. $f(x) = x(x - 3)$

x-intercepts: $(0, 0), (3, 0)$

$f'(x) - 2x - 3 = 0$ at $x = \dfrac{3}{2}$.

6. $f(x) = -3x\sqrt{x + 1}$

x-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2} = -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}.$$

8. $f(x) = x^2 - 5x + 4, \; [1, 4]$

$f(1) = f(4) = 0$

f is continuous on $[1, 4]$. f is differentiable on $(1, 4)$.

Rolle's Theorem applies.

$$f'(x) = 2x - 5$$

$$2x - 5 = 0 \qquad x = \frac{5}{2}$$

c value: $\frac{5}{2}$

10. $f(x) = (x - 3)(x + 1)^2, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$\begin{aligned} f'(x) &= (x - 3)(2)(x + 1) + (x + 1)^2 \\ &= (x + 1)[2x - 6 + x + 1] \\ &= (x + 1)(3x - 5) \end{aligned}$$

c value: $\frac{5}{3}$

12. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

14. $f(x) = \dfrac{x^2 - 1}{x}, \; [-1, 1]$

$f(-1) = f(1) = 0$

f is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = \cos x, \; [0, 2\pi]$

$f(0) = f(2\pi) = 1$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = -\sin x$$

c value: π

18. $f(x) = \cos 2x, \; \left[-\dfrac{\pi}{12}, \dfrac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{0}\right)$$

Rolle's Theorem does not apply.

20. $f(x) = \sec x, \; \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

$$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

f is continuous on $[-\pi/4, \pi/4]$. f is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$f'(x) = \sec x \tan x$$

$$\sec x \tan x = 0$$

$$x = 0$$

c value: 0

22. $f(x) = x - x^{1/3}, \; [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

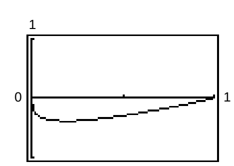
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

c value: $\dfrac{\sqrt{3}}{9} \approx 0.1925$



24. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6},\ [-1, 0]$

$f(-1) = f(0) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$. Rolle's Theorem applies.

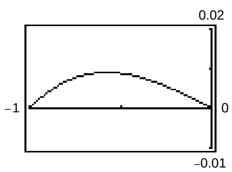
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi}\quad [\text{Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

c value: -0.5756



26. $C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$

(a) $C(3) = C(6) = \frac{25}{3}$

(b) $C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

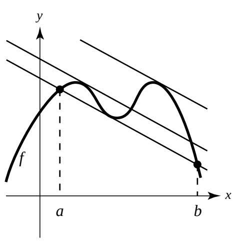
$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

In the interval $(3, 6)$: $c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098$.

28.



30. $f(x) = |x - 3|,\ [0, 6]$

f is not differentiable at $x = 3$.

32. $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

34. $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

36. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$f'(x) = 3x^2 = 1$$

$$x = \pm\frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \dfrac{\sqrt{3}}{3}$.

38. $f(x) = 2\sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = 2\cos x + 2\cos 2x = 0$$

$$2[\cos x + 2\cos^2 x - 1] = 0$$

$$2(2\cos x - 1)(\cos x + 1) = 0$$

$$\cos x = \frac{1}{2}$$

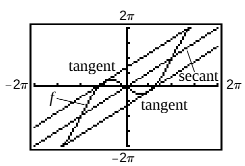
$$\cos x = -1$$

$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \dfrac{\pi}{3}$.

40. $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

(a)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$\cos x = 0$

$$c = \pm\frac{\pi}{2}, \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\dfrac{\pi}{2} - 2\right) = 1\left(x - \dfrac{\pi}{2}\right)$

$$y = x - 2$$

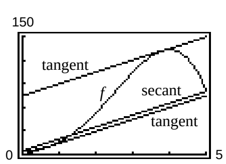
$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

42. $f(x) = -x^4 + 4x^3 + 8x^2 + 5$, $(0, 5)$, $(5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$

(a)



(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

(c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

44. $S(t) = 200\left(5 - \dfrac{9}{2+t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \dfrac{450}{7}$

(b) $S'(t) = 200\left(\dfrac{9}{(2+t)^2}\right) = \dfrac{450}{7}$

$$\frac{1}{(2+t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

46. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \qquad g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$

$g\left(\dfrac{a}{k}\right) = g\left(\dfrac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\dfrac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\dfrac{a}{k}, \dfrac{b}{k}\right]$

Critical number of g: $\dfrac{c}{k}$

48. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

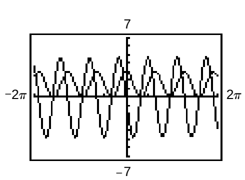
$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/hr.}$$

By the Mean Value Theorem, there exists a time to, $0 < t_0 < 5$, such that $T'(t_0) = -222$.

50. $f(x) = 3\cos^2\left(\dfrac{\pi x}{2}\right), \quad f'(x) = 6\cos\left(\dfrac{\pi x}{2}\right)\left(-\sin\left(\dfrac{\pi x}{2}\right)\right)\left(\dfrac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)



(b) f and f' are both continuous on the entire real line.

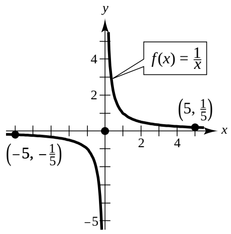
(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

(d) $\lim_{x \to 3^-} f'(x) = 0$

$\lim_{x \to 3^+} f'(x) = 0$

52. f is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



54. False. f must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

56. True

58. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

60. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

62. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \quad\quad |b - a| \text{ since } |-\sin c| \quad\quad 1.$$

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

2. $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$

Decreasing on: $(-1, \infty)$

4. $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0), (1, \infty)$

Decreasing on: $(-\infty, -1), (0, 1)$

6. $y = \dfrac{x^2}{x + 1}$

$y' = \dfrac{x(x + 2)}{(x + 1)^2}$

Critical numbers: $x = 0, -2$ Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -2$	$-2 < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on $(-\infty, -2), (0, \infty)$

Decreasing on $(-2, -1), (-1, 0)$

8. $h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$

$h'(x) = 0$

Critical numbers: $x = \pm 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-3, 3)$

Decreasing on $(-\infty, -3), (3, \infty)$

10. $y = x + \dfrac{4}{x}$

$y' = \dfrac{(x - 2)(x + 2)}{x^2}$

Critical numbers: $x = \pm 2$ Discontinuity: 0

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing: $(-\infty, -2), (2, \infty)$

Decreasing: $(-2, 0), (0, 2)$

12. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-4, \infty)$

Decreasing on: $(-\infty, -4)$

Relative minimum: $(-4, -6)$

14. $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, -4)$

Decreasing on: $(-4, \infty)$

Relative maximum: $(-4, 4)$

16. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on $(-\infty, 0), (4, \infty)$

Decreasing on $(0, 4)$

Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

18. $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (0, \infty)$

Decreasing on: $(-2, 0)$

Relative maximum: $(-2, 0)$

Relative minimum: $(0, -4)$

20. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

Relative minimum: $(2, -44)$

22. $f(x) = x^{2/3} - 4$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

Relative minimum: $(0, -4)$

24. $f(x) = (x - 1)^{1/3}$

$f'(x) = \frac{1}{3(x - 1)^{2/3}}$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

26. $f(x) = |x + 3| - 1$

$f'(x) = \frac{x + 3}{|x + 3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

Relative minimum: $(-3, -1)$

28. $f(x) = \frac{x}{x + 1}$

$f'(x) = \frac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \frac{1}{(x + 1)^2}$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

30. $f(x) = \frac{x+3}{x^2} = \frac{1}{x} + \frac{3}{x^2}$

$f'(x) = -\frac{1}{x^2} - \frac{6}{x^3} = \frac{-(x+6)}{x^3}$

Critical number: $x = -6$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\frac{1}{12}\right)$

32. $f(x) = \frac{x^2 - 3x - 4}{x - 2}$

$f'(x) = \frac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2} = \frac{x^2 - 4x + 10}{(x-2)^2}$

Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

34. $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x,\ 0 < x < 2\pi$

$f'(x) = \cos 2x = 0$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Relative maxima: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, -\frac{1}{2}\right), \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$

36. $f(x) = \dfrac{\sin x}{1 + \cos^2 x}, \; 0 < x < 2\pi$

$f'(x) = \dfrac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

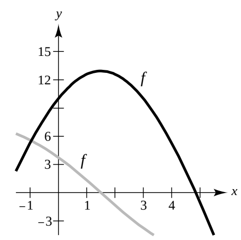
Relative maximum: $\left(\dfrac{\pi}{2}, 1\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -1\right)$

38. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)



(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

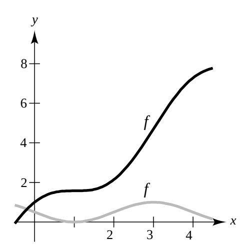
$\left(0, \frac{3}{2}\right)$	$\left(\frac{3}{2}, 5\right)$
$f'(x) > 0$	$f'(x) < 0$
Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

40. $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2}$

(b)



(c) $\dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

$(0, \pi)$	$(\pi, 4\pi)$
$f'(x) > 0$	$f'(x) > 0$
Increasing	Increasing

f is increasing when f' is positive.

42. $f(t) = \cos^2 t - \sin^2 t = -2\sin^2 t = g(t),\ -2 < t < 2$

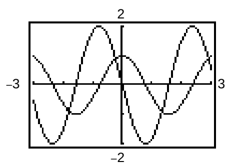
$f'(t) = -4\sin t\cos t = -2\sin 2t$

f symmetric with respect to y-axis

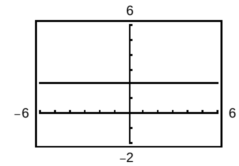
zeros of f: $\pm\frac{\pi}{4}$

Relative maximum: $(0, 1)$

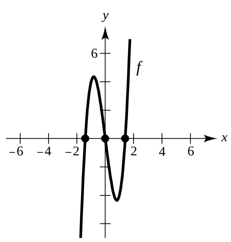
Relative minimum: $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right)$



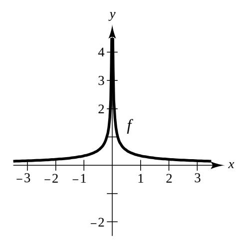
44. $f(x)$ is a line of slope ≈ 2 $\quad f'(x) = 2.$



46. f is a 4th degree polynomial $\quad f'$ is a cubic polynomial.



48. f has positive slope



In Exercises 50–54, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

50. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

52. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

54. $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

56. Critical number: $x = 5$

$f'(4) = -2.5 \qquad f$ is decreasing at $x = 4$.

$f'(6) = 3 \qquad f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

58. $s(t) = 4.9(\sin\theta)t^2$

(a) $v(t) = 9.8(\sin\theta)t \qquad$ speed $= |9.8(\sin\theta)t|$

(b) If $\theta = \pi/2$, the speed is maximum,

$v(t) = 9.8t.$

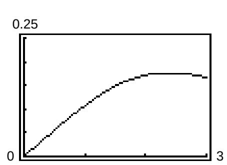
60. $C = \frac{3t}{27 + t^3}, t \quad 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.

(b)



The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \frac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$

$= \frac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

By the First Derivative Test, this is a maximum.

62. $P = 2.44x - \dfrac{x^2}{20{,}000} - 5000, 0 \quad x \quad 35{,}000$

$P' = 2.44 - \dfrac{x}{10{,}000} = 0$

$x = 24{,}400$

Test intervals:	$0 < x < 24{,}400$	$24{,}400 < x < 35{,}000$
Sign of P':	$P' > 0$	$P' < 0$

Increasing when $0 < x < 24{,}400$ hamburgers.

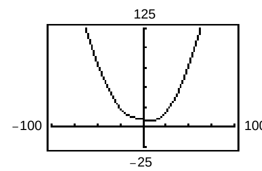
Decreasing when $24{,}400 < x < 35{,}000$ hamburgers.

64. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

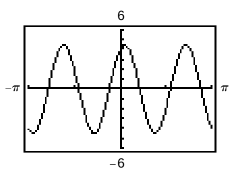
$T = 10°, R \approx 8.3666\Omega$

(b)



The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10°$.

66. $f(x) = 2\sin(3x) + 4\cos(3x)$



The maximum value is approximately 4.472. You could use calculus by finding $f'(x)$ and then observing that the maximum value of f occurs at a point where $f'(x) = 0$. For instance, $f'(0.154) \approx 0$, and $f(0.154) = 4.472$.

68. (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

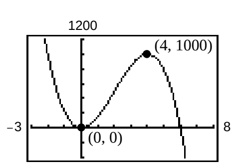
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

$(0, 0)$:	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
$(4, 1000)$:	$1000 = 64a_3 + 16a_2$	$(f(4) = 1000)$
	$0 = 48a_3 + 8a_2$	$(f'(4) = 0)$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = \dfrac{375}{2}$, $a_3 = \dfrac{-125}{4}$

$$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$$

(d)



70. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(1, 2)$: $2 = a_4 + a_3 + a_2 + a_1 + a_0$ $(f(1) = 2)$

$0 = 4a_4 + 3a_3 + 2a_2 + a_1$ $(f'(1) = 0)$

$(-1, 4)$: $4 = a_4 - a_3 + a_2 - a_1 + a_0$ $(f(-1) = 4)$

$0 = -4a_4 + 3a_3 - 2a_2 + a_1$ $(f'(-1) = 0)$

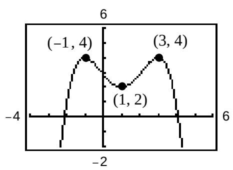
$(3, 4)$: $4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ $(f(3) = 4)$

$0 = 108a_4 + 27a_3 + 6a_2 + a_1$ $(f'(3) = 0)$

(c) The solution is $a_0 = \frac{23}{8}, a_1 = -\frac{3}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{2}, a_4 = -\frac{1}{8}$

$f(x) = -\frac{1}{8}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{3}{2}x + \frac{23}{8}$.

(d)



72. False

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

74. True

If $f(x)$ is an nth-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

76. False.

The function might not be continuous.

78. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f.

Section 3.4 Concavity and the Second Derivative Test

2. $y = -x^3 + 3x^2 - 2, y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

4. $f(x) = \dfrac{x^2 - 1}{2x + 1}, y'' = \dfrac{-6}{(2x + 1)^3}$

Concave upward: $\left(-\infty, -\frac{1}{2}\right)$

Concave downward: $\left(-\frac{1}{2}, \infty\right)$

6. $y = \dfrac{1}{270}(-3x^5 + 40x^3 + 135x), y'' = \dfrac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

10. $y = x + 2\csc x, \quad (-\pi, \pi)$

$y' = 1 - 2\csc x \cot x$

$y'' = -2\csc x(-\csc^2 x) - 2\cot x(-\csc x \cot x)$

$= 2(\csc^3 x + \csc x \cot^2 x)$

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

12. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.

Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

14. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$ when $x = 0$.

However, $(0, 3)$ is not a point of inflection since $f''(x) \quad 0$ for all x.

Concave upward on $(-\infty, \infty)$

16. $f(x) = x^3(x - 4)$

$f'(x) = x^3 + 3x^2(x - 4)$

$= x^2[x + 3(x - 4)] = 4x^2(x - 3)$

$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$

$f''(x) = 12x(x - 2) = 0$ when $x = 0, 2$.

Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0), (2, -16)$

18. $f(x) = x\sqrt{x + 1}$, Domain: $[-1, \infty)$

$$f'(x) = (x)\frac{1}{2}(x + 1)^{-1/2} + \sqrt{x + 1} = \frac{3x + 2}{2\sqrt{x + 1}}$$

$$f''(x) = \frac{6\sqrt{x + 1} - (3x + 2)(x + 1)^{-1/2}}{4(x + 1)} = \frac{3x + 4}{4(x + 1)^{3/2}}$$

$f''(x) > 0$ on the entire domain of f (except for $x = -1$, for which $f''(x)$ is undefined).
There are no points of inflection.

Concave upward on $(-1, \infty)$

20. $f(x) = \dfrac{x + 1}{\sqrt{x}}$ Domain: $x > 0$

$$f'(x) = \frac{x - 1}{2x^{3/2}}$$

$$f''(x) = \frac{3 - x}{4x^{5/2}}$$

Point of inflection: $\left(3, \dfrac{4}{\sqrt{3}}\right) = \left(3, \dfrac{4\sqrt{3}}{3}\right)$

Test intervals	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

22. $f(x) = 2\csc\dfrac{3x}{2},\ 0 < x < 2\pi$

$f'(x) = -3\csc\dfrac{3x}{2}\cot\dfrac{3x}{2}$

$f''(x) = \dfrac{9}{2}\left(\csc^3\dfrac{3x}{2} + \csc\dfrac{3x}{2}\cot^2\dfrac{3x}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $\left(0, \dfrac{2\pi}{3}\right), \left(\dfrac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

No points of inflection

24. $f(x) = \sin x + \cos x,\ 0 \quad x \quad 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0$ when $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$.

Test interval:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\dfrac{3\pi}{4}, 0\right), \left(\dfrac{7\pi}{4}, 0\right)$

26. $f(x) = x + 2\cos x,\ [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{3\pi}{2}$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

28. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

30. $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, $(5, 0)$ is a relative maximum.

32. $f(x) = x^3 - 9x^2 + 27x$

$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$

$f''(x) = 6(x - 3)$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \quad 0$ for all x and, therefore, there are no relative extrema.

34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

$g'(x) = \frac{-(x - 4)(x - 1)(x + 2)}{2}$

$g''(x) = 3 + 3x - \frac{3}{2}x^2$

Critical numbers: $x = -2, 1, 4$

$g''(-2) = -9 < 0$

$(-2, 0)$ is a relative maximum.

$g''(1) = 9/2 > 0$

$(1, -10.125)$ is a relative minimum.

$g''(4) = -9 < 0$

$(4, 0)$ is a relative maximum.

36. $f(x) = \sqrt{x^2 + 1}$

$f'(x) = \frac{x}{\sqrt{x^2 + 1}}$

Critical number: $x = 0$

$f''(x) = \frac{1}{(x^2 + 1)^{3/2}}$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

38. $f(x) = \frac{x}{x - 1}$

$f'(x) = \frac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

40. $f(x) = 2 \sin x + \cos 2x, 0 \quad x \quad 2\pi$

$f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x - 4 \sin x \cos x = 2 \cos x(1 - 2 \sin x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$f''(x) = -2 \sin x - 4 \cos 2x$

$f''\left(\frac{\pi}{6}\right) < 0$

$f''\left(\frac{\pi}{2}\right) > 0$

$f''\left(\frac{5\pi}{6}\right) < 0$

$f''\left(\frac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

42. $f(x) = x^2\sqrt{6 - x^2}, [-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

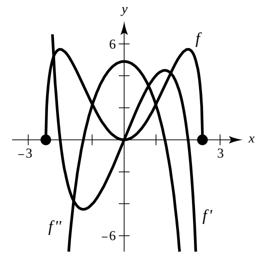
$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0$ $(0, 0)$ is a relative minimum.

$f''(\pm 2) < 0$ $(\pm 2, 4\sqrt{2})$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c) 

The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

44. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \dfrac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$f''(x) = -\sqrt{2x}\sin x + \dfrac{\cos x}{\sqrt{2x}} + \dfrac{\cos x}{\sqrt{2x}} - \dfrac{\sin x}{2x\sqrt{2x}}$

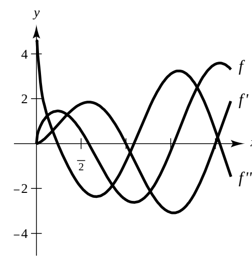
$= \dfrac{2\cos x}{\sqrt{2x}} - \dfrac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$

$= \dfrac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$

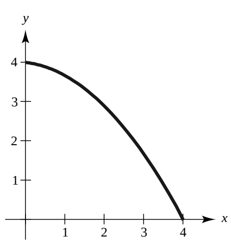
(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

Points of inflection: $(0.75, 0.83), (3.42, -0.72)$

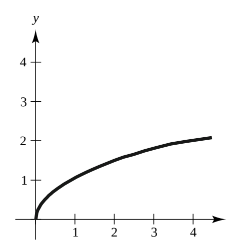
(c) 

f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

46. (a) 

$f' < 0$ means f decreasing

f' decreasing means concave downward

(b) 

$f' > 0$ means f increasing

f' decreasing means concave downward

48. (a) The rate of change of sales is increasing.
$S'' > 0$

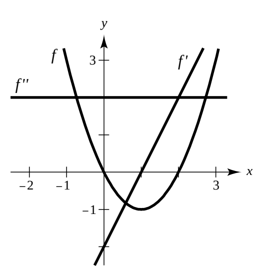
(b) The rate of change of sales is decreasing.
$S' > 0, S'' < 0$

(c) The rate of change of sales is constant.
$S' = C, S'' = 0$

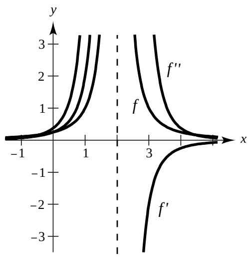
(d) Sales are steady.
$S = C, S' = 0, S'' = 0$

(e) Sales are declining, but at a lower rate.
$S' < 0, S'' > 0$

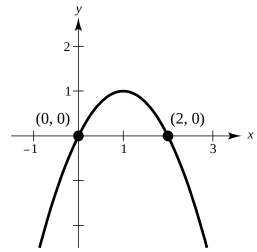
(f) Sales have bottomed out and have started to rise.
$S' > 0$

50. 

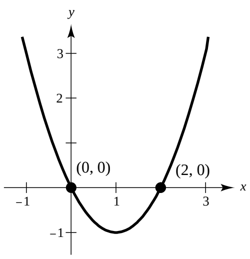
52.



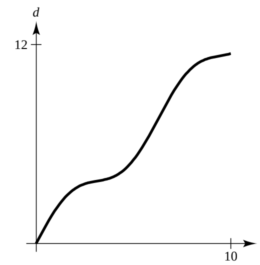
54.



56.



58. (a)



(b) Since the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

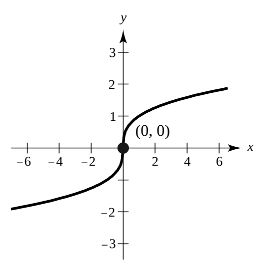
60. (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \qquad 28a + 6b + c = -1$$

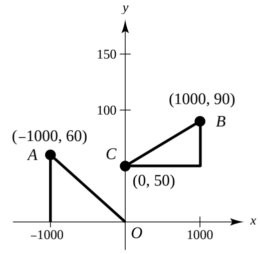
$f'(2) = 12a + 4b + c = 0, \quad f'(4) = 48a + 8b + c = 0, \quad f''(3) = 18a + 2b = 0$

$$\begin{array}{llll} 28a + 6b + c = -1 & & 18a + 2b = 0 \\ 12a + 4b + c = 0 & & 16a + 2b = -1 \\ 16a + 2b = -1 & & 2a = 1 \end{array}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

64. (a) line OA: $y = -0.06x$ slope: -0.06



line CB: $y = 0.04x + 50$ slope: 0.04

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

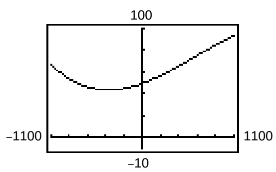
$-0.06 = (1000)^2\, 3a - 2000b + c$

$(1000, 90)$: $90 = (1000)^3 a + (1000)^2 b + 1000c + d$

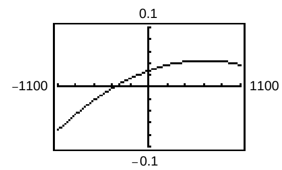
$0.04 = (1000)^2\, 3a + 2000b + c$

The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)

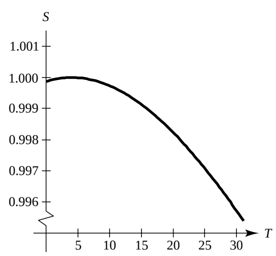


(d) The steepest part of the road is 6% at the point A.

66. $S = \dfrac{5.755T^3}{10^8} - \dfrac{8.521T^2}{10^6} + \dfrac{0.654T}{10^4} + 0.99987, \quad 0 < T < 25$

(a) The maximum occurs when $T \approx 4°$ and $S \approx 0.999999$.

(b)



(c) $S(20°) \approx 0.9982$

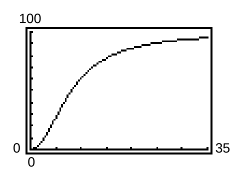
68. $C = 2x + \dfrac{300{,}000}{x}$

$C' = 2 - \dfrac{300{,}000}{x^2} = 0$ when $x = 100\sqrt{15} \approx 387$

By the First Derivative Test, C is minimized when $x \approx 387$ units.

70. $S = \dfrac{100t^2}{65 + t^2}, \; t > 0$

(a)



(b) $S'(t) = \dfrac{13{,}000t}{(65 + t^2)^2}$

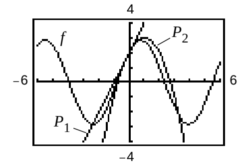
$S''(t) = \dfrac{13{,}000(65 - 3t^2)}{(65 + t^2)^3} = 0 \qquad t = 4.65$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As t increases, the speed increases, but at a slower rate.

72. $f(x) = 2(\sin x + \cos x), \qquad f(0) = 2$



$f'(x) = 2(\cos x - \sin x), \qquad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x), \qquad f''(0) = -2$

$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$

$P_1'(x) = 2$

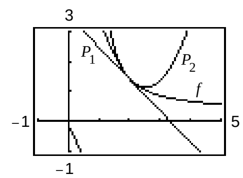
$P_2(x) = 2 + 2(x - 0) + \frac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$

$P_2'(x) = 2 - 2x$

$P_2''(x) = -2$

The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

74. $f(x) = \dfrac{\sqrt{x}}{x - 1}, \qquad f(2) = \sqrt{2}$



$f'(x) = \dfrac{-(x + 1)}{2\sqrt{x}(x - 1)^2}, \qquad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2 + 6x - 1}{4x^{3/2}(x - 1)^3}, \qquad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x - 2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x - 2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x - 2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x - 2) + \dfrac{23\sqrt{2}}{32}(x - 2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x - 2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

76. $f(x) = x(x - 6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x - 2)(x - 6) = 0$

$f''(x) = 6x - 24 = 6(x - 4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of f.

78. $p(x) = ax^3 + bx^2 + cx + d$

$p'(x) = 3ax^2 + 2bx + c$

$p''(x) = 6ax + 2b$

$6ax + 2b = 0$

$x = -\frac{b}{3a}$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$

$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

80. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

82. True

$y = \sin(bx)$

Slope: $y' = b\cos(bx)$

$-b \quad y' \quad b$ (Assume $b > 0$)

84. False. For example, let $f(x) = (x - 2)^4$.

Section 3.5 Limits at Infinity

2. $f(x) = \frac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

4. $f(x) = 2 + \frac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

6. $f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1}$

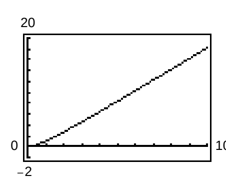
No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e)

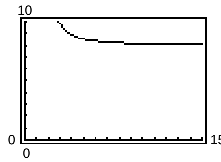
8. $f(x) = \frac{2x^2}{x + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998



$\lim_{x \quad \infty} f(x) = \infty$ (Limit does not exist.)

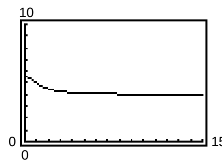
10. $f(x) = \dfrac{8x}{\sqrt{x^2 - 3}}$



x	10^1	10^2	10^3	10^4	10^5	10^6	10^7
$f(x)$	8.12	8.001	8	8	8	8	8

$\lim_{x \to \infty} f(x) = 8$

12. $f(x) = 4 + \dfrac{3}{x^2 + 2}$



x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4.0	4.0	4	4

$\lim_{x \to \infty} f(x) = 4$

14. (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{5x^2 - 3x + 7}{x} = 5x - 3 + \dfrac{7}{x}$

$\lim_{x \to \infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^2 - 3x + 7}{x^2} = 5 - \dfrac{3}{x} + \dfrac{7}{x^2}$

$\lim_{x \to \infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^2 - 3x + 7}{x^3} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{7}{x^3}$

$\lim_{x \to \infty} h(x) = 0$

16. (a) $\lim_{x \to \infty} \dfrac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x \to \infty} \dfrac{3 - 2x}{3x - 1} = -\dfrac{2}{3}$

(c) $\lim_{x \to \infty} \dfrac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist)

18. (a) $\lim_{x \to \infty} \dfrac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x \to \infty} \dfrac{5x^{3/2}}{4x^{3/2} + 1} = \dfrac{5}{4}$

(c) $\lim_{x \to \infty} \dfrac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist)

20. $\lim_{x \to \infty} \dfrac{3x^3 + 2}{9x^3 - 2x^2 + 7} = \dfrac{3}{9} = \dfrac{1}{3}$

22. $\lim_{x \to \infty} \left(4 + \dfrac{3}{x}\right) = 4 + 0 = 4$

24. $\lim_{x \to \infty} \left(\dfrac{1}{2}x - \dfrac{4}{x^2}\right) = -\infty$ (Limit does not exist)

26. $\lim_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}} = \lim_{x \to -\infty} \dfrac{1}{\left(\dfrac{\sqrt{x^2 + 1}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

$= \lim_{x \to -\infty} \dfrac{-1}{\sqrt{x + (1/x)}} = -1$

28. $\lim_{x \to -\infty} \frac{-3x + 1}{\sqrt{x^2 + x}} = \lim_{x \to -\infty} \frac{-3 + (1/x)}{\frac{\sqrt{x^2 + x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } -\sqrt{x^2} = x\right)$

$$= \lim_{x \to -\infty} \frac{3 - (1/x)}{\sqrt{1 + (1/x)}} = 3$$

30. $\lim_{x \to \infty} \frac{x - \cos x}{x} = \lim_{x \to \infty} \left(1 - \frac{\cos x}{x}\right)$

$$= 1 - 0 = 1$$

Note:

$\lim_{x \to \infty} \frac{\cos x}{x} = 0$ by the Squeeze Theorem since

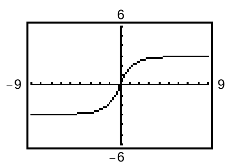
$-\frac{1}{x} \quad \frac{\cos x}{x} \quad \frac{1}{x}$.

32. $\lim_{x \to \infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$

34. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

$\lim_{x \to \infty} f(x) = 3$

$\lim_{x \to -\infty} f(x) = -3$



Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

36. $\lim_{x \to \infty} x \tan \frac{1}{x} = \lim_{t \to 0^+} \frac{\tan t}{t} = \lim_{t \to 0^+} \left[\frac{\sin t}{t} \cdot \frac{1}{\cos t}\right]$

$$= (1)(1) = 1$$

(Let $x = 1/t$.)

38. $\lim_{x \to \infty} \left(2x - \sqrt{4x^2 + 1}\right) = \lim_{x \to \infty} \left[\left(2x - \sqrt{4x^2 + 1}\right) \cdot \frac{2x + \sqrt{4x^2 + 1}}{2x + \sqrt{4x^2 + 1}}\right] = \lim_{x \to \infty} \frac{-1}{2x + \sqrt{4x^2 + 1}} = 0$

40. $\lim_{x \to -\infty} \left(3x + \sqrt{9x^2 - x}\right) = \lim_{x \to -\infty} \left[\left(3x + \sqrt{9x^2 - x}\right) \cdot \frac{3x - \sqrt{9x^2 - x}}{3x - \sqrt{9x^2 - x}}\right]$

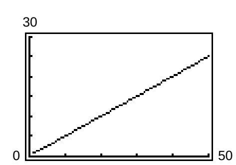
$$= \lim_{x \to -\infty} \frac{x}{3x - \sqrt{9x^2 - x}}$$

$$= \lim_{x \to -\infty} \frac{1}{3 - \frac{\sqrt{9x^2 - x}}{-\sqrt{x^2}}} \quad \left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$$

$$= \lim_{x \to -\infty} \frac{1}{3 + \sqrt{9 - (1/x)}} = \frac{1}{6}$$

42.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1



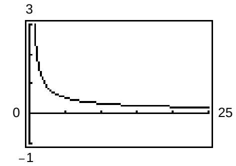
$\lim_{x \to \infty} \frac{x^2 - x\sqrt{x^2 - x}}{1} \cdot \frac{x^2 + x\sqrt{x^2 - x}}{x^2 + x\sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{x^3}{x^2 + x\sqrt{x^2 - x}} = \infty$

Limit does not exist.

44.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001

$$\lim_{x \to \infty} \frac{x+1}{x\sqrt{x}} = 0$$



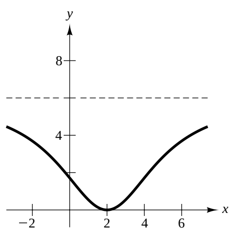
46. $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

$f'(x) > 0$ for $x > 2$.

$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 6$

For example, let $f(x) = \dfrac{-6}{0.1(x-2)^2 + 1} + 6$.



48. (a) The function is even: $\lim_{x \to -\infty} f(x) = 5$

(b) The function is odd: $\lim_{x \to -\infty} f(x) = -5$

50. $y = \dfrac{x-3}{x-2}$

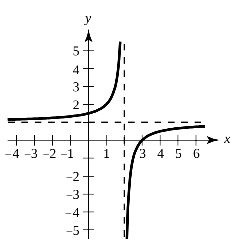
Intercepts: $(3, 0), \left(0, \dfrac{3}{2}\right)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x-3}{x-2} = 1 = \lim_{x \to \infty} \frac{x-3}{x-2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



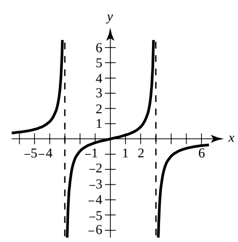
52. $y = \dfrac{2x}{9 - x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



54. $y = \dfrac{x^2}{x^2 - 9}$

Intercept: $(0, 0)$

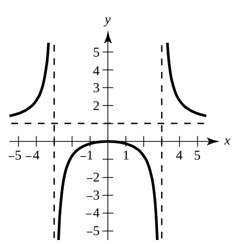
Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x^2}{x^2 - 9} = 1 = \lim_{x \to \infty} \frac{x^2}{x^2 - 9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



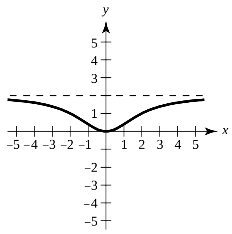
56. $y = \dfrac{2x^2}{x^2 + 4}$

Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Relative minimum: $(0, 0)$



58. $x^2y = 4$

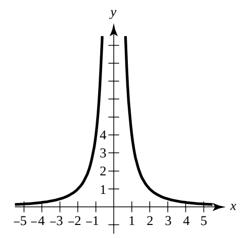
Intercepts: none

Symmetry: y-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x\to-\infty} \frac{4}{x^2} = 0 = \lim_{x\to\infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



60. $y = \dfrac{2x}{1 - x^2}$

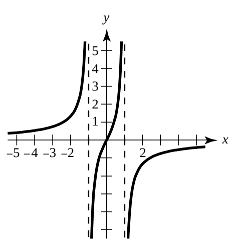
Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$ since

$$\lim_{x\to-\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x\to\infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)



62. $y = 1 + \dfrac{1}{x}$

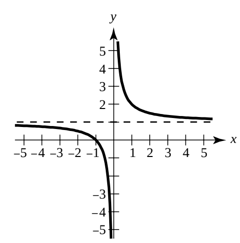
Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right) = 1 = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



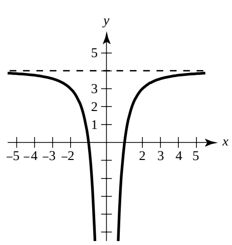
64. $y = 4\left(1 - \dfrac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$



66. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

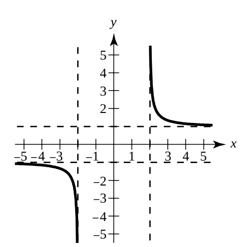
Intercepts: none

Symmetry: origin

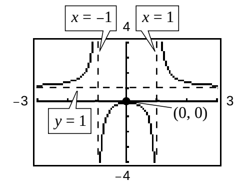
Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \ \lim_{x\to-\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)



68. $f(x) = \dfrac{x^2}{x^2 - 1}$



$$f'(x) = \frac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2} = 0 \text{ when } x = 0.$$

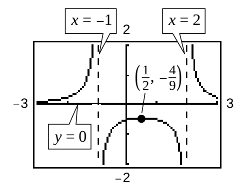
$$f''(x) = \frac{(x^2-1)^2(-2) + 2x(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{2(3x^2+1)}{(x^2-1)^3}$$

Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$

70. $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x+1)(x-2)}$



$$f'(x) = \frac{-(2x-1)}{(x^2-x-2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

$$f''(x) = \frac{(x^2-x-2)^2(-2) + (2x-1)(2)(x^2-x-2)(2x-1)}{(x^2-x-2)^4}$$

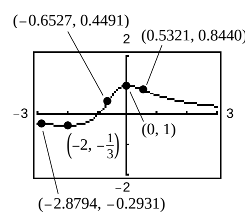
$$= \frac{6(x^2-x+1)}{(x^2-x-2)^3}$$

Since $f''\left(\frac{1}{2}\right) < 0$, then $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$

72. $f(x) = \dfrac{x+1}{x^2 + x + 1}$



$$f'(x) = \frac{-x(x+2)}{(x^2+x+1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3+3x^2-1)}{(x^2+x+1)^3} = 0 \text{ when } x \approx 0.5321, -0.6527, -2.8794.$$

$f''(0) < 0$

Therefore, $(0, 1)$ is a relative maximum.

$f''(-2) > 0$

Therefore,

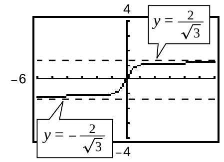
$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$

74. $g(x) = \dfrac{2x}{\sqrt{3x^2 + 1}}$



$g'(x) = \dfrac{2}{(3x^2 + 1)^{3/2}}$

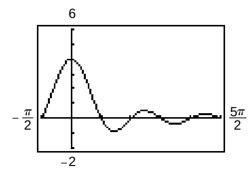
$g''(x) = \dfrac{-18x}{(3x^2 + 1)^{5/2}}$

No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm\dfrac{2}{\sqrt{3}}$

No vertical asymptotes

76. $f(x) = \dfrac{2 \sin 2x}{x}$ Hole at $(0, 4)$



$f'(x) = \dfrac{4x \cos 2x - 2 \sin 2x}{x^2}$

There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

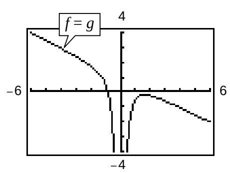
Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes

78. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)

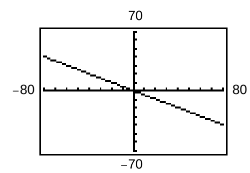


(b) $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$= -\left[\dfrac{x^3}{2x^2} - \dfrac{2x^2}{2x^2} + \dfrac{2}{2x^2}\right]$

$= -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2} = g(x)$

(c)

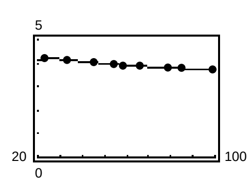


The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

80. $\displaystyle\lim_{v_1/v_2 \to \infty} 100\left[1 - \frac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

82. $y = \dfrac{3.351t^2 + 42.461t - 543.730}{t^2}$

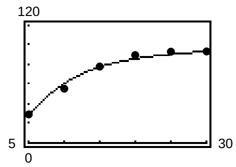
(a)



(b) Yes. $\displaystyle\lim_{t \to \infty} y = 3.351$

84. $S = \dfrac{100t^2}{65 + t^2},\ t > 0$

(a)



(b) Yes. $\lim_{t \to \infty} S = \dfrac{100}{1} = 100$

86. $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m.

Case 1: If $n < m$: $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{\dfrac{a_n}{x^{m-n}} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{0}{b_m} = 0$

Case 2: If $m = n$: $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{a_n + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{a_n}{b_m}.$

Case 3: If $n > m$: $\lim_{x \to \infty} \dfrac{p(x)}{q(x)} = \lim_{x \to \infty} \dfrac{a_n x^{n-m} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty.$

88. False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. Thus, $y_1' = 1/\left(2\sqrt{x+1}\right)$ and $y_1'(0) = 1/2$. Finally,

$y_1'' = -\dfrac{1}{4(x+1)^{3/2}}$ and $y_1''(0) = -\dfrac{1}{4}.$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$. Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \ge 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/\left(2\sqrt{x+1}\right), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x, but $f(x)$ *increases* without bound.

Section 3.6 A Summary of Curve Sketching

2. The slope of f approaches ∞ as $x \quad 0^-$, and approaches $-\infty$ as $x \quad 0^+$. Matches (C)

4. The slope is positive up to approximately $x = 1.5$. Matches (B)

6. (a) x_0, x_2, x_4

(b) x_2, x_3

(c) x_1

(d) x_1

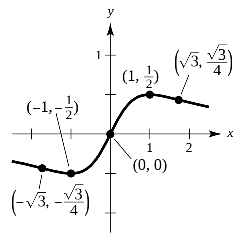
(e) x_2, x_3

8. $y = \dfrac{x}{x^2 + 1}$

$$y' = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0 \text{ when } x = \pm 1.$$

$$y'' = -\frac{2x(3 - x^2)}{(x^2 + 1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

Horizontal asymptote: $y = 0$



	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\frac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\frac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up

10. $y = \dfrac{x^2 + 1}{x^2 - 9}$

$y' = \dfrac{-20x}{(x^2 - 9)^2} = 0$ when $x = 0$

$y'' = \dfrac{60(x^2 + 3)}{(x^2 - 9)^3} < 0$ when $x = 0$

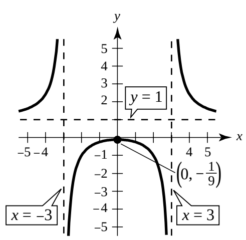
Therefore, $\left(0, -\dfrac{1}{9}\right)$ is a relative maximum.

Intercept: $\left(0, -\dfrac{1}{9}\right)$

Vertical asymptotes: $x = \pm 3$

Horizontal asymptote: $y = 1$

Symmetric about y-axis



12. $f(x) = \dfrac{x + 2}{x} = 1 + \dfrac{2}{x}$

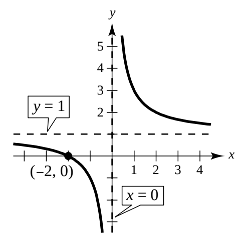
$f'(x) = \dfrac{-2}{x^2} < 0$ when $x \neq 0$.

$f''(x) = \dfrac{4}{x^3} \neq 0$

Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$



14. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x - 4)(x^2 + 4x + 16)}{x^3} = 0$ when $x = 4$.

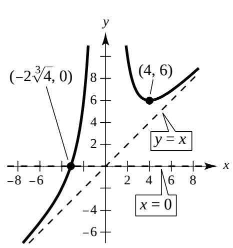
$f''(x) = \dfrac{192}{x^4} > 0$ if $x \neq 0$.

Therefore, $(4, 6)$ is a relative minimum.

Intercept; $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



16. $f(x) = \dfrac{x^3}{x^2 - 4} = x + \dfrac{4x}{x^2 - 4}$

$f'(x) = \dfrac{x^2(x^2 - 12)}{(x^2 - 4)^2} = 0$ when $x = 0, \pm 2\sqrt{3}$

$f''(x) = \dfrac{8x(x^2 + 12)}{(x^2 - 4)^3} = 0$ when $x = 0$

Intercept: $(0, 0)$

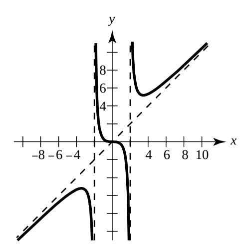
Relative maximum: $\left(-2\sqrt{3}, -3\sqrt{3}\right)$

Relative minimum: $\left(2\sqrt{3}, 3\sqrt{3}\right)$

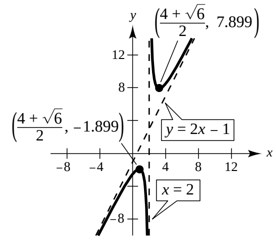
Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$



18. $y = \dfrac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \dfrac{3}{x - 2}$



$y' = 2 - \dfrac{3}{(x - 2)^2} = \dfrac{2x^2 - 8x + 5}{(x - 2)^2} = 0$ when $x = \dfrac{4 \pm \sqrt{6}}{2}$.

$y'' = \dfrac{6}{(x - 2)^3} \neq 0$

Relative maximum: $\left(\dfrac{4 - \sqrt{6}}{2}, -1.8990\right)$

Relative minimum: $\left(\dfrac{4 + \sqrt{6}}{2}, 7.8990\right)$

Intercept: $(0, -5/2)$

Vertical asymptote: $x = 2$

Slant asymptote: $y = 2x - 1$

20. $g(x) = x\sqrt{9 - x}$ Domain: x 9

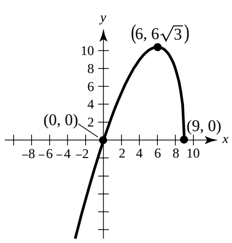
$g'(x) = \dfrac{3(6 - x)}{2\sqrt{9 - x}} = 0$ when $x = 6$

$g''(x) = \dfrac{3(x - 12)}{4(9 - x)^{3/2}} < 0$ when $x = 6$

Relative maximum: $(6, 6\sqrt{3})$

Intercepts: $(0, 0)$, $(9, 0)$

Concave downward on $(-\infty, 9)$



22. $y = x\sqrt{16 - x^2}$ Domain: -4 x 4

$y' = \dfrac{2(8 - x^2)}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$

$y'' = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}} = 0$ when $x = 0$

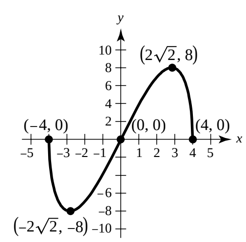
Relative maximum: $(2\sqrt{2}, 8)$

Relative minimum: $(-2\sqrt{2}, -8)$

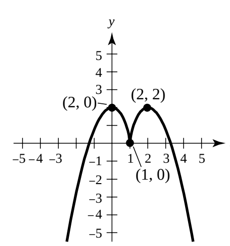
Intercepts: $(0, 0)$, $(\pm 4, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$



24. $y = 3(x - 1)^{2/3} - (x - 1)^2$



$y' = \dfrac{2}{(x - 1)^{1/3}} - 2(x - 1) = \dfrac{2 - 2(x - 1)^{4/3}}{(x - 1)^{1/3}} = 0$ when $x = 0, 2$

(y' undefined for $x = 1$)

$y'' = \dfrac{-2}{3(x - 1)^{4/3}} - 2 < 0$ for all $x \neq 1$

Concave downward on $(-\infty, 1)$ and $(1, \infty)$

Relative maximum: $(0, 2)$, $(2, 2)$

Relative minimum: $(1, 0)$

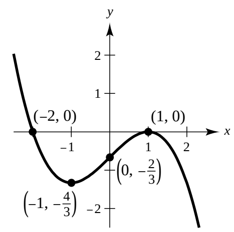
Intercepts: $(0, 2)$, $(1, 0)$, $(-1.280, 0)$, $(3.280, 0)$

26. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$

$y'' = -2x = 0$ when $x = 0$

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

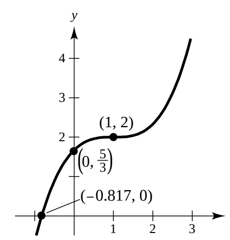


28. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

$f'(x) = (x - 1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x - 1) = 0$ when $x = 1$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	2	0	0	Point of inflection
$1 < x < \infty$		$+$	$+$	Increasing, concave up



30. $f(x) = (x + 1)(x - 2)(x - 5)$

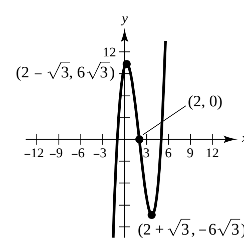
$f'(x) = (x + 1)(x - 2) + (x + 1)(x - 5) + (x - 2)(x - 5)$

$= 3(x^2 - 4x + 1) = 0$ when $x = 2 \pm \sqrt{3}$.

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 2 - \sqrt{3}$		$+$	$-$	Increasing, concave down
$x = 2 - \sqrt{3}$	$6\sqrt{3}$	0	$-$	Relative maximum
$2 - \sqrt{3} < x < 2$		$-$	$-$	Decreasing, concave down
$x = 2$	0	$-$	0	Point of inflection
$2 < x < 2 + \sqrt{3}$		$-$	$+$	Decreasing, concave up
$x = 2 + \sqrt{3}$	$-6\sqrt{3}$	0	$+$	Relative minimum
$2 + \sqrt{3} < x < \infty$		$+$	$+$	Increasing, concave up

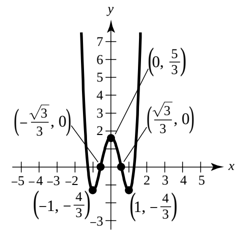
Intercepts: $(0, 10), (-1, 0), (2, 0), (5, 0)$



32. $y = 3x^4 - 6x^2 + \frac{5}{3}$

$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0$ when $x = 0, x = \pm 1$.

$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$.

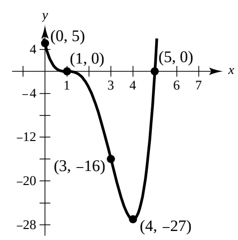


	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-4/3$	0	$+$	Relative minimum
$-1 < x < -\frac{\sqrt{3}}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{\sqrt{3}}{3}$	0	$+$	0	Point of inflection
$-\frac{\sqrt{3}}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	$5/3$	0	$-$	Relative maximum
$0 < x < \frac{\sqrt{3}}{3}$		$-$	$-$	Decreasing, concave down
$x = \frac{\sqrt{3}}{3}$	0	$-$	0	Point of inflection
$\frac{\sqrt{3}}{3} < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	$-4/3$	0	$+$	Relative minimum
$1 < x < \infty$		$+$	$+$	Increasing, concave up

34. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

$f'(x) = 4x^3 - 24x^2 + 36x - 16 = 4(x - 4)(x - 1)^2 = 0$ when $x = 1, x = 4$.

$f''(x) = 12x^2 - 48x + 36 = 12(x - 3)(x - 1) = 0$ when $x = 3, x = 1$.

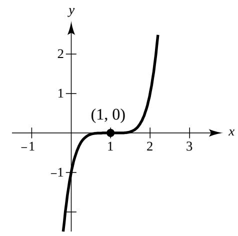


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	0	0	0	Point of inflection
$1 < x < 3$		$-$	$-$	Decreasing, concave down
$x = 3$	-16	$-$	0	Point of inflection
$3 < x < 4$		$-$	$+$	Decreasing, concave up
$x = 4$	-27	0	$+$	Relative minimum
$4 < x < \infty$		$+$	$+$	Increasing, concave up

36. $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.



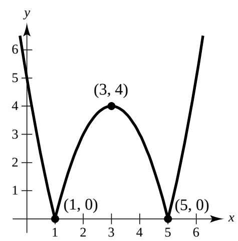
	y	y'	y''	Conclusion
$-\infty < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		$+$	$+$	Increasing, concave up

38. $y = |x^2 - 6x + 5|$

$$y' = \frac{2(x-3)(x^2-6x+5)}{|x^2-6x+5|} = \frac{2(x-3)(x-5)(x-1)}{|(x-5)(x-1)|}$$

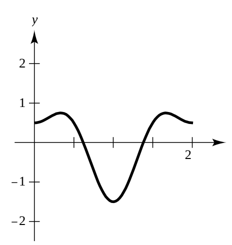
$= 0$ when $x = 3$ and undefined when $x = 1$, $x = 5$.

$$y'' = \frac{2(x^2-6x+5)}{|x^2-6x+5|} = \frac{2(x-5)(x-1)}{|(x-5)(x-1)|} \text{ undefined when } x = 1,\ x = 5.$$



	y	y'	y''	Conclusion
$-\infty < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		$+$	$-$	Increasing, concave down
$x = 3$	4	0	$-$	Relative maximum
$3 < x < 5$		$-$	$-$	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		$+$	$+$	Increasing, concave up

40. $y = \cos x - \frac{1}{2}\cos 2x,\ 0 \quad x \quad 2\pi$



$y' = -\sin x + \sin 2x = -\sin x(1 - 2\cos x) = 0$ when $x = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$.

$y'' = -\cos x + 2\cos 2x = -\cos x + 2(2\cos^2 x - 1)$

$= 4\cos^2 x - \cos x - 2 = 0$ when $\cos x = \frac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931$.

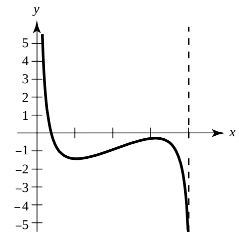
Therefore, $x \approx 0.5678$ or 5.7154, $x \approx 2.2057$ or 4.0775.

Relative maxima: $\left(\frac{\pi}{3}, \frac{3}{4}\right), \left(\frac{5\pi}{3}, \frac{3}{4}\right)$

Relative minimum: $\left(\pi, -\frac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323), (2.2057, -0.4449), (5.7154, 0.6323), (4.0775, -0.4449)$

42. $y = 2(x - 2) + \cot x,\ 0 < x < \pi$



$y' = 2 - \csc^2 x = 0$ when $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

$y'' = 2\csc^2 x \cot x = 0$ when $x = \dfrac{\pi}{2}$

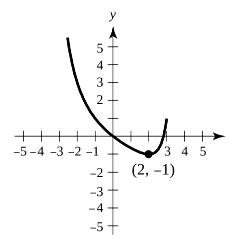
Relative maximum: $\left(\dfrac{3\pi}{4}, \dfrac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 3\right)$

Point of inflection: $\left(\dfrac{\pi}{2}, \pi - 4\right)$

Vertical asymptotes: $x = 0, \pi$

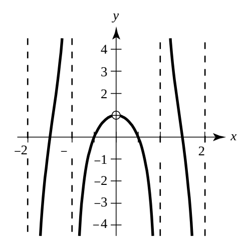
44. $y = \sec^2\left(\dfrac{\pi x}{8}\right) - 2\tan\left(\dfrac{\pi x}{8}\right) - 1,\ -3 < x < 3$



$y' = 2\sec^2\left(\dfrac{\pi x}{8}\right)\tan\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) - 2\sec^2\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) = 0 \qquad x = 2$

Relative minimum: $(2, -1)$

46. $g(x) = x\cot x,\ -2\pi < x < 2\pi$



$g'(x) = \dfrac{\sin x \cos x - x}{\sin^2 x}$

$g'(0)$ does not exist. But $\lim_{x \to 0} x \cot x = \lim_{x \to 0} \dfrac{x}{\tan x} = 1$.

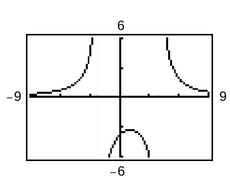
Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

Intercepts: $\left(-\dfrac{3\pi}{2}, 0\right), \left(-\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{2}, 0\right)$

Symmetric with respect to y-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$

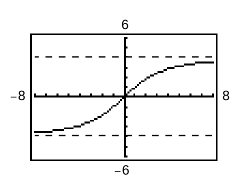
48. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$



$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

50. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$



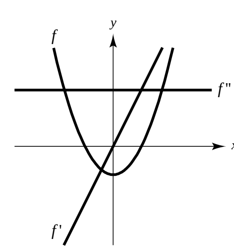
$y = \pm 4$ horizontal asymptotes

$(0, 0)$ point of inflection

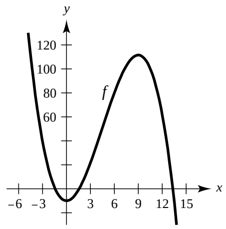
52. f'' is constant.

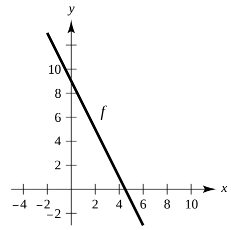
f' is linear.

f is quadratic.



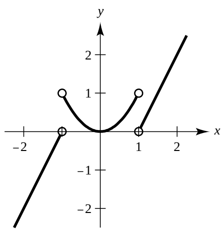
54.

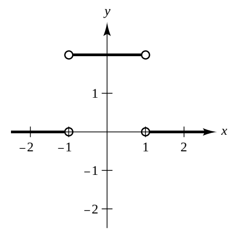




(any vertical translate of f will do)

56.





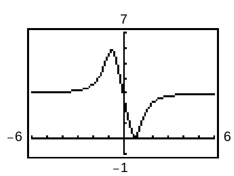
(any vertical translate of the 3 segments of f will do)

58. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

60. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

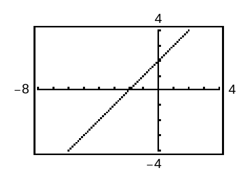


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

62. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

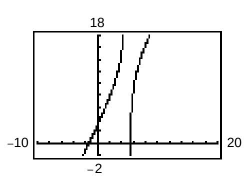
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



hole at $(1, 3)$

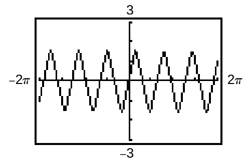
64. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



The graph appears to approach the slant asymptote $y = 2x + 2$.

66. $f(x) = \tan(\sin \pi x)$

(a)



(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$= -\tan(\sin \pi x) = -f(x)$

Symmetry with repect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of f is concave downward.

68. Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x+3}$$

70. Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1-x^2}{x}$$

72. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax-2), a \neq 0$

$f'(x) = a^2x - a = a(ax-1) = 0$ when $x = \frac{1}{a}$.

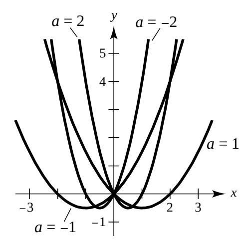
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

(b)



74. Tangent line at P: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $(0, f(x_0) - x_0 f'(x_0))$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0) f'(x_0)$$

x-intercept: $(x_0 + f(x_0) f'(x_0), 0)$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = {y_0}^2 + \left(\frac{f(x_0)}{f'(x_0)}\right)^2 = \frac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC|^2 = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = |x_0 - (x_0 + f(x_0) f'(x_0))| = |f(x_0) f'(x_0)|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + {y_0}^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

Section 3.7 Optimization Problems

2. Let x and y be two positive numbers such that $x + y = S$.

$$P = xy = x(S - x) = Sx - x^2$$

$$\frac{dP}{dx} = S - 2x = 0 \text{ when } x = \frac{S}{2}.$$

$$\frac{d^2P}{dx^2} = -2 < 0 \text{ when } x = \frac{S}{2}.$$

P is a maximum when $x = y = S/2$.

4. Let x and y be two positive numbers such that $xy = 192$.

$$S = x + 3y = \frac{192}{y} + 3y$$

$$\frac{dS}{dy} = 3 - \frac{192}{y^2} = 0 \text{ when } y = 8.$$

$$\frac{d^2S}{dy^2} = \frac{384}{y^3} > 0 \text{ when } y = 8.$$

S is minimum when $y = 8$ and $x = 24$.

6. Let x and y be two positive numbers such that $x + 2y = 100$.

$$P = xy = y(100 - 2y) = 100y - 2y^2$$

$$\frac{dP}{dy} = 100 - 4y = 0 \text{ when } y = 25.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 25.$$

P is a maximum when $x = 50$ and $y = 25$.

8. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$

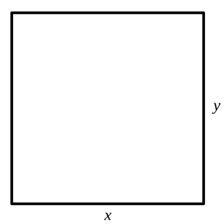
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)



10. Let x be the length and y the width of the rectangle.

$$xy = A$$

$$y = \frac{A}{x}$$

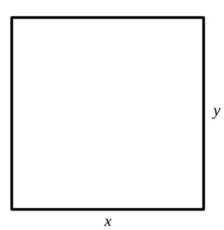
$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

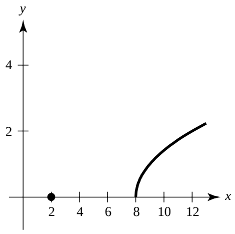
P is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)



12. $f(x) = \sqrt{x-8}$, $(2, 0)$

From the graph, it is clear that $(8, 0)$ is the closest point on the graph of f to $(2, 0)$.



14. $f(x) = (x+1)^2$, $(5, 3)$

$$\begin{aligned} d &= \sqrt{(x-5)^2 + [(x+1)^2 - 3]^2} \\ &= \sqrt{x^2 - 10x + 25 + (x^2 + 2x - 2)^2} \\ &= \sqrt{x^2 - 10x + 25 + x^4 + 4x^3 - 8x + 4} \\ &= \sqrt{x^4 + 4x^3 + x^2 - 18x + 29} \end{aligned}$$

Since d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$\begin{aligned} g(x) &= x^4 + 4x^3 + x^2 - 18x + 29 \\ g'(x) &= 4x^3 + 12x^2 + 2x - 18 \\ &= 2(x-1)(2x^2 + 8x + 9) \end{aligned}$$

By the First Derivative Test, $x = 1$ yields a minimum. Hence, $(1, 4)$ is closest to $(5, 3)$.

16. $F = \dfrac{v}{22 + 0.02v^2}$

$$\frac{dF}{dv} = \frac{22 - 0.02v^2}{(22 + 0.02v^2)^2}$$

$$= 0 \text{ when } v = \sqrt{1100} \approx 33.166.$$

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

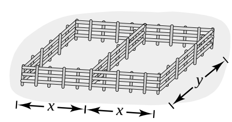
18. $4x + 3y = 200$ is the perimeter. (see figure)

$$A = 2xy = 2x\left(\frac{200 - 4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(50 - 2x) = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 25.$$

A is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.



20. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

(b) $V = x(24 - 2x)^2$, $0 < x < 12$

(c) $$\frac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$$

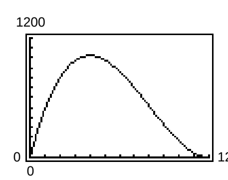
$$= 12(12 - x)(4 - x) = 0 \text{ when } x = 12, 4 \text{ (12 is not in the domain).}$$

$$\frac{d^2V}{dx^2} = 12(2x - 16)$$

$$\frac{d^2V}{dx^2} < 0 \text{ when } x = 4.$$

When $x = 4$, $V = 1024$ is maximum.

(d)



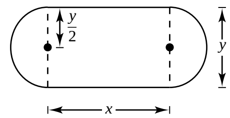
The maximum volume seems to be 1024.

22. (a) $P = 2x + 2\pi r$

$= 2x + 2\pi\left(\frac{y}{2}\right)$

$= 2x + \pi y = 200$

$y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$



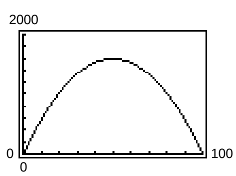
(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) = 1528$

The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

(e)



Maximum area is approximately

1591.55 m^2 $(x = 50 \text{ m})$.

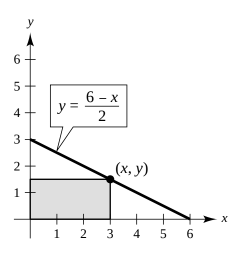
24. You can see from the figure that $A = xy$ and $y = \frac{6 - x}{2}$.

$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2)$.

$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0$ when $x = 3$.

$\frac{d^2A}{dx^2} = -1 < 0$ when $x = 3$.

A is a maximum when $x = 3$ and $y = 3/2$.



26. (a) $A = \frac{1}{2}\text{ base} \times \text{height}$

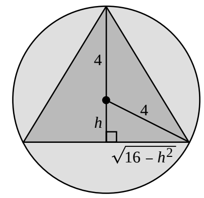
$$= \frac{1}{2}\left(2\sqrt{16 - h^2}\right)(4 + h)$$

$$= \sqrt{16 - h^2}(4 + h)$$

$$\frac{dA}{dh} = \frac{1}{2}(16 - h^2)^{-1/2}(-2h)(4 + h) + (16 - h^2)^{1/2}$$

$$= (16 - h^2)^{-1/2}\left[-h(4 + h) + (16 - h^2)\right]$$

$$= \frac{-2(h^2 + 2h - 8)}{\sqrt{16 - h^2}} = \frac{-2(h + 4)(h - 2)}{\sqrt{16 - h^2}}$$

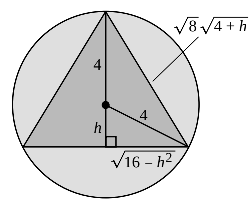


$\frac{dA}{dh} = 0$ when $h = 2$, which is a maximum by the First Derivative Test.

Hence, the sides are $2\sqrt{16 - h^2} = 4\sqrt{3}$, an equilateral triangle. Area $= 12\sqrt{3}$ sq. units.

(b) $\cos\alpha = \frac{4 + h}{\sqrt{8}\sqrt{4 + h}} = \frac{\sqrt{4 + h}}{\sqrt{8}}$

$$\tan\alpha = \frac{\sqrt{16 - h^2}}{4 + h}$$



$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{16 - h^2}\right)(4 + h)$$

$$= (4 + h)^2 \tan\alpha$$

$$= 64\cos^4\alpha\tan\alpha$$

$$A'(\alpha) = 64\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$$

$$\cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$$

$$1 = 4\cos\alpha\sin\alpha\tan\alpha$$

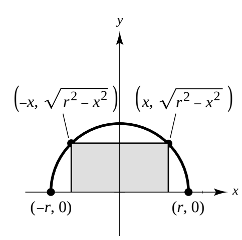
$$\frac{1}{4} = \sin^2\alpha$$

$$\sin\alpha = \frac{1}{2} \qquad \alpha = 30^\circ \text{ and } A = 12\sqrt{3}.$$

(c) Equilateral triangle

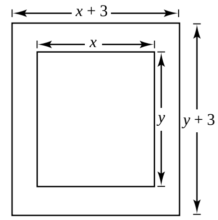
28. $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$



By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $\left(\sqrt{2}r\right)/2$.

30. $xy = 36 \qquad y = \dfrac{36}{x}$



$$A = (x + 3)(y + 3) = (x + 3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \qquad 3x^2 = 108 \qquad x = 6, y = 6$$

Dimensions: 9×9

32. $V = \pi r^2 h = V_0$ cubic units or $h = \dfrac{V_0}{\pi r^2}$

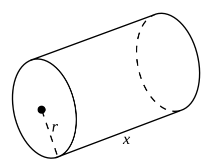
$$S = 2\pi r^2 + 2\pi rh = 2\left(\pi r^2 + \frac{V_0}{r}\right)$$

$$\frac{dS}{dr} = 2\left(2\pi r - \frac{V_0}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{V_0}{2\pi}} \text{ units.}$$

$$h = \frac{V_0}{\pi\left(\sqrt[3]{V_0/2\pi}\right)^2} = \frac{V_0(2\pi)^{2/3}}{\pi V_0^{2/3}} = \frac{2V_0^{1/3}}{(2\pi)^{1/3}} = 2r$$

By the First Derivative Test, this will yield the minimum surface area.

34. $V = \pi r^2 x$



$x + 2\pi r = 108 \qquad x = 108 - 2\pi r \quad$ (see figure)

$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

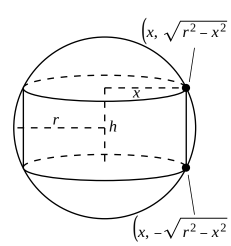
$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.

36. $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right) = 2\pi x^2\sqrt{r^2 - x^2} \quad$ (see figure)



$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \qquad x = \frac{\sqrt{6}r}{3}.$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$

38. No. The volume will change because the shape of the container changes when squeezed.

40. $V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

42. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, \quad 0 \quad x \quad \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, \quad 0 \quad x \quad 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20 - 5x}{6}\right)^2, 0 \quad x \quad 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \quad x \quad \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

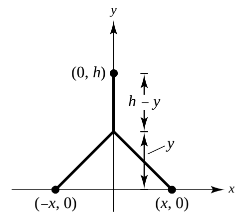
44. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

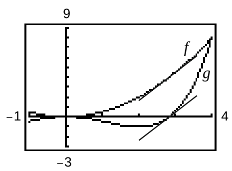
$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



46. $f(x) = \frac{1}{2}x^2$ $\qquad g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$

(a)



(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$

$d'(x) = 2x - \frac{1}{4}x^3 = 0 \qquad 8x = x^3$

$x = 0, 2\sqrt{2}$ (in $[0, 4]$)

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

(c) $f'(x) = x$, Tangent line at $\left(2\sqrt{2}, 4\right)$ is

$$y - 4 = 2\sqrt{2}\left(x - 2\sqrt{2}\right)$$
$$y = 2\sqrt{2}x - 4.$$

$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $\left(2\sqrt{2}, 0\right)$ is

$$y - 0 = \left(\tfrac{1}{4}\left(2\sqrt{2}\right)^3 - 2\sqrt{2}\right)\left(x - 2\sqrt{2}\right)$$
$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

(d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point x where the distance is maximum.

48. Let F be the illumination at point P which is x units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d - x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d - x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d - x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d - x}$$

$$(d - x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

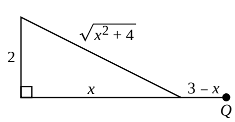
$$d\sqrt[3]{I_1} = x\left(\sqrt[3]{I_1} + \sqrt[3]{I_2}\right)$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d - x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

50. (a) $T = \dfrac{\sqrt{x^2 + 4}}{2} + \dfrac{(3 - x)}{4}$



(b) $$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2 + 4}} - \frac{1}{4} = 0$$

$$\frac{x}{\sqrt{x^2 + 4}} = \frac{1}{2}$$

$$2x^2 = x^2 + 4$$

$$x^2 = 4$$

$$x = 2$$

$$T(2) = \sqrt{2} + \frac{1}{4} \text{ hours}$$

—CONTINUED—

50. —CONTINUED—

(c) $$T = \frac{\sqrt{x^2+4}}{v_1} + \frac{(3-x)}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+4}} - \frac{1}{v_2} = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \frac{v_1}{v_2}$$

$$\sin\theta = \frac{v_1}{v_2}$$

θ depends on $\frac{v_1}{v_2}$ only.

(d) $$\text{Cost} = \sqrt{x^2+4}\,C_1 + (3-x)C_2 = \frac{\sqrt{x^2+4}}{(1/C_1)} + \frac{(3-x)}{(1/C_2)}$$

From above, $\sin\theta = \dfrac{1/C_1}{1/C_2} = \dfrac{C_2}{C_1}$

52. $$T = \frac{\sqrt{x^2+d_1^2}}{v_1} + \frac{\sqrt{d_2^2+(a-x)^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+d_1^2}} + \frac{x-a}{v_2\sqrt{d_2^2+(a-x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2+d_1^2}} = \sin\theta_1 \text{ and } \frac{x-a}{\sqrt{d_2^2+(a-x)^2}} = -\sin\theta_2$$

we have

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \qquad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2+d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2+(a-x)^2]^{3/2}} > 0$$

this condition yields a minimum time.

54. $$C(x) = 2k\sqrt{x^2+4} + k(4-x)$$

$$C'(x) = \frac{2xk}{\sqrt{x^2+4}} - k = 0$$

$$2x = \sqrt{x^2+4}$$

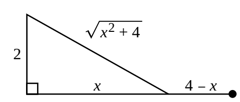
$$4x^2 = x^2+4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \dfrac{C_2}{C_1} = \dfrac{1}{2}$ $\qquad \theta = 30°$.

Thus, $x = \dfrac{2}{\sqrt{3}}$.



56. $$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144-r^2}$$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144-r^2)^{-1/2}(-2r) + 2r\sqrt{144-r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r-3r^3}{\sqrt{144-r^2}}\right] = \pi\left[\frac{r(96-r^2)}{\sqrt{144-r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone: $S = \pi\left(4\sqrt{6}\right)\sqrt{\left(4\sqrt{6}\right)^2 + \left(4\sqrt{3}\right)^2} = 48\sqrt{6}\pi$

Area of sector: $144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}\left(3-\sqrt{6}\right) \approx 1.153 \text{ radians or } 66°$$

58. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$P = (0.12)d - id$

$d = ki^2$ (since d is proportional to i^2)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$$\frac{dP}{di} = k(0.24i - 3i^2) = 0 \text{ when } i = \frac{0.24}{3} = 0.08.$$

$$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0 \text{ when } i = 0.08 \text{ (**Note:** } k > 0).$$

The profit is a maximum when $i = 8\%$.

60. $P = -\frac{1}{10}s^3 + 6s^2 + 400$

(a) $\frac{dP}{ds} = -\frac{3}{10}s^2 + 12s = -\frac{3}{10}s(s - 40) = 0$ when $x = 0$, $s = 40$.

$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12$

$\frac{d^2P}{ds^2}(0) > 0$ $s = 0$ yields a minimum.

$\frac{d^2P}{ds^2}(40) < 0$ $s = 40$ yields a maximum.

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3{,}600{,}000$).

(b) $\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

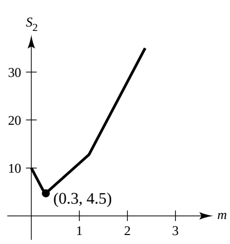
The point of diminishing returns occurs when $s = 20$, which corresonds to \$20,000 being spent on advertising.

62. $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

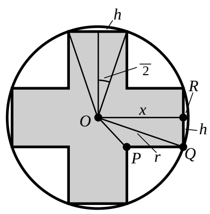
Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7$ mi.



64. (a) Label the figure so that $r^2 = x^2 + h^2$.
Then, the area A is 8 times the area of the region given by $OPQR$:



$$A = 8\left[\frac{1}{2}h^2 + (x - h)h\right]$$

$$= 8\left[\frac{1}{2}(r^2 - x^2) + \left(x - \sqrt{r^2 - x^2}\right)\sqrt{r^2 - x^2}\right]$$

$$= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$

$$A'(x) = 8\sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0$$

$$\frac{8x^2}{\sqrt{r^2 - x^2}} = 8x + 8\sqrt{r^2 - x^2}$$

$$x^2 = x\sqrt{r^2 - x^2} + (r^2 - x^2)$$

$$2x^2 - r^2 = x\sqrt{r^2 - x^2}$$

$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2 - x^2)$$

$$5x^4 - 5x^2r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$$x = r\sqrt{\frac{5 + \sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

(b) Note that $\sin\frac{\theta}{2} = \frac{h}{r}$ and $\cos\frac{\theta}{2} = \frac{x}{r}$. The area A of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2$$

$$= 8r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2}$$

$$= 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$

$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\theta$$

$$\tan\theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4°$$

(c) Note that $x^2 = \frac{r^2}{10}\left(5 + \sqrt{5}\right)$ and $r^2 - x^2 = \frac{r^2}{10}\left(5 - \sqrt{5}\right)$.

$$A(x) = 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$

$$= 8\left[\frac{r^2}{10}\left(5 + \sqrt{5}\right)\frac{r^2}{10}\left(5 - \sqrt{5}\right)\right]^{1/2} + 4\frac{r^2}{10}\left(5 + \sqrt{5}\right) - 4r^2$$

$$= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2$$

$$= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2$$

$$= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right)$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$.

Thus, $A(\theta) = 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$

$$= 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right)$$

$$= \frac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right)$$

Section 3.8 Newton's Method

2. $f(x) = 2x^2 - 3$

$f'(x) = 4x$

$x_1 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	4	$-\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{5}{4} = 1.25$	0.125	5.0	0.025	1.225

4. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

6. $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

Approximation of the zero of f is 0.755.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	-0.4688	1.3125	-0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

8. $f(x) = x - 2\sqrt{x+1}$

$f'(x) = 1 - \frac{1}{\sqrt{x+1}}$

Approximation of the zero of f is 4.8284.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	5	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	.00085	4.8284

10. $f(x) = 1 - 2x^3$

$f'(x) = -6x^2$

Approximation of the zero of f is 0.7937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	-1	-6	0.1667	0.8333
2	0.8333	-0.1573	-4.1663	0.0378	0.7955
3	0.7955	-0.0068	-3.7969	0.0018	0.7937
4	0.7937	0.0000	-3.7798	0.0000	0.7937

12. $f(x) = \frac{1}{2}x^4 - 3x - 3$

$f'(x) = 2x^3 - 3$

Approximation of the zero of f is -0.8937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	0.5	-5	-0.1	-0.9
2	-0.9	0.0281	-4.458	-0.0063	-0.8937
3	-0.8937	0.0001	-4.4276	0.0000	-0.8937

Approximation of the zero of f is 2.0720.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	-1	13	-0.0769	2.0769
2	2.0769	0.0725	14.9175	0.0049	2.0720
3	2.0720	-0.0003	14.7910	0.0000	2.0720

14. $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

Approximation of the zero of f is 0.866.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0001	3.0087	0.0000	0.8655

16. $h(x) = f(x) - g(x) = 3 - x - \frac{1}{x^2 + 1}$

$h'(x) = -1 + \frac{2x}{(x^2 + 1)^2}$

Point of intersection of the graphs of f and g occurs when $x \approx 2.893$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	2.9000	-0.0063	-0.9345	0.0067	2.8933
2	2.8933	0.0000	-0.9341	0.0000	2.8933

18. $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

One point of intersection of the graphs of f and g occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the y-axis, the other point of intersection occurs when $x \approx -0.824$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.8000	-0.0567	2.3174	-0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

20. $f(x) = x^n - a = 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{x_i^n - a}{nx_i^{n-1}}$$

$$= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

22. $x_{i+1} = \dfrac{x_i^2 + 5}{2x_i}$

i	1	2	3	4
x_i	2.0000	2.2500	2.2361	2.2361

$\sqrt{5} \approx 2.236$

24. $x_{i+1} = \dfrac{2x_i^3 + 15}{3x_i^2}$

i	1	2	3	4
x_i	2.5000	2.4667	2.4662	2.4662

$\sqrt[3]{15} \approx 2.466$

26. $f(x) = \tan x$

$f'(x) = \sec^2 x$

Approximation of the zero: 3.142

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	3.0000	−0.1425	1.0203	−0.1397	3.1397
2	3.1397	−0.0019	1.0000	−0.0019	3.1416
3	3.1416	0.0000	1.0000	0.0000	3.1416

28. $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$y' = 12x^2 - 24x + 12 = f'(x)$

$x_1 = \dfrac{3}{2}$

$f'(x_2) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{1}{2}$	1
2	1	1	0	—	—

30. $f(x) = 2\sin x + \cos 2x$

$f'(x) = 2\cos x - 2\sin 2x$

$x_1 = \dfrac{3\pi}{2}$

Fails because $f'(x_1) = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
1	$\frac{3\pi}{2}$	−3	0

32. Newton's Method could fail if $f'(c) \approx 0$, or if the initial value x_1 is far from c.

34. Let $g(x) = f(x) - x = \cot x - x$

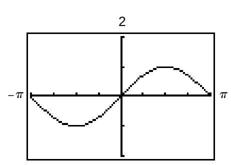
$g'(x) = -\csc^2 x - 1.$

The fixed point is approximately 0.86.

n	x_n	$g(x_n)$	$g'(x_n)$	$\dfrac{g(x_n)}{g'(x_n)}$	$x_n - \dfrac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.3579	−2.4123	0.1484	0.8516
2	0.8516	0.0240	−2.7668	−0.0087	0.8603
3	0.8603	0.0001	−2.7403	0.0000	0.8603

36. $f(x) = \sin x,\ f'(x) = \cos x$

(a)



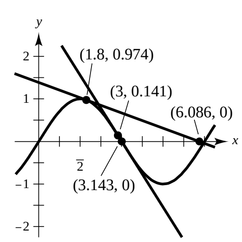
(b) $x_1 = 1.8$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 6.086$

(c) $x_1 = 3$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \approx 3.143$

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

38. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$\frac{1}{3} \approx 0.333$

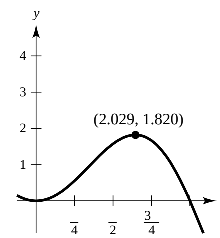
(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$\frac{1}{11} \approx 0.091$

40. $f(x) = x \sin x,\ (0, \pi)$

$f'(x) = x \cos x + \sin x = 0$



Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

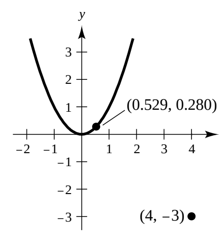
$[F'(x) = 2 \cos x - x \sin x]$

n	x_n	$F(x_n)$	$F'(x_n)$	$\frac{F(x_n)}{F'(x_n)}$	$x_n - \frac{F(x_n)}{F'(x_n)}$
1	2.0000	0.0770	-2.6509	-0.0290	2.0290
2	2.0290	-0.0007	-2.7044	0.0002	2.0288

Approximation to the critical number: 2.029

42. $y = f(x) = x^2,\ (4, -3)$

$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$



d is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

$g(x) = D' = 4x^3 + 14x - 8$

$g'(x) = 12x^2 + 14$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	0.5000	-0.5000	17.0000	-0.0294	0.5294
2	0.5294	0.0051	17.3632	0.0003	0.5291
3	0.5291	-0.0001	17.3594	0.0000	0.5291

$x \approx 0.529$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.

44. Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

46. $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843, 1 \quad x \quad 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248.$

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	−5.0750	37.7240	−0.1345	1.1345
2	1.1345	−0.2805	33.5849	−0.0084	1.1429
3	1.1429	0.0006	33.3293	0.0000	1.1429

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.5000	4.6725	−24.8760	−0.1878	3.6878
2	3.6878	−0.3286	−28.3550	0.0116	3.6762
3	3.6762	−0.0009	−28.1450	0.0000	3.6762

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

48. True

50. True

52. $f(x) = \sqrt{4 - x^2}\sin(x - 2)$

Domain: $[-2, 2]$

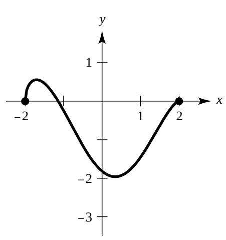
$x = -2$ and $x = 2$ are both zeros.

$f'(x) = \sqrt{4 - x^2}\cos(x - 2) - \frac{x}{\sqrt{4 - x^2}}\sin(x - 2)$

Let $x_1 = -1$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	−1.0000	−0.2444	−1.7962	0.1361	−1.1361
2	−1.1361	−0.0090	−1.6498	0.0055	−1.1416
3	−1.1416	0.0000	−1.6422	0.0000	−1.1416

Zeros: $x = \pm 2, x \approx -1.142$



Section 3.9 Differentials

2. $f(x) = \dfrac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \dfrac{-12}{x^3}$

Tangent line at $\left(2, \dfrac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \dfrac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$	1.65	1.515	1.5	1.485	1.35

4. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $(2, \sqrt{2})$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \dfrac{x}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

6. $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1.0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

8. $y = f(x) = 1 - 2x^2,\ f'(x) = -4x,\ x = 0,\ \Delta x = dx = -0.1$

$$\Delta y = f(x + \Delta x) - f(x) = f(-0.1) - f(0) = [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02$$

$$dy = f'(x)\,dx = f'(0)(-0.1) = (0)(-0.1) = 0$$

10. $y = f(x) = 2x + 1,\ f'(x) = 2,\ x = 2,\ \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x) = f(2.01) - f(2) = [2(2.01) + 1] - [2(2) + 1] = 0.02$$

$$dy = f'(x)\,dx = f'(2)(0.01) = 2(0.01) = 0.02$$

12. $y = 3x^{2/3}$

$$dy = 2x^{-1/3}dx = \frac{2}{x^{1/3}}dx$$

14. $y = \sqrt{9 - x^2}$

$$dy = \frac{1}{2}(9 - x^2)^{-1/2}(-2x)dx = \frac{-x}{\sqrt{9 - x^2}}\,dx$$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$$dy = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}\right)dx = \frac{x - 1}{2x\sqrt{x}}\,dx$$

18. $y = x \sin x$

$$dy = (x \cos x + \sin x)\,dx$$

20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{(x^2 + 1)2 \sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2}\right]dx$$

$$= \left[\frac{2 \sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2}\right]dx$$

22. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + (-1)(-0.1) = 1.1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + (-1)(0.04) = 0.96$$

24. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + 0(-0.1) = 1$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + 0(0.04) = 1$$

26. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + (3)(-0.07) = 7.79$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + (3)(0.1) = 8.3$$

28. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + 5(-0.07) = 7.65$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + 5(0.1) = 8.5$$

30. $A = \frac{1}{2}bh,\ b = 36,\ h = 50$

$db = dh = \pm 0.25$

$dA = \frac{1}{2}b\,dh + \frac{1}{2}h\,db$

$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ square centimeters

32. $x = 12$ inches

$\Delta x = dx = \pm 0.03$ inch

(a) $V = x^3$

$dV = 3x^2\,dx = 3(12)^2(\pm 0.03)$

$= \pm 12.96$ cubic inches

(b) $S = 6x^2$

$dS = 12x\,dx = 12(12)(\pm 0.03)$

$= \pm 4.32$ square inches

34. (a) $C = 56$ centimeters

$\Delta C = dC = \pm 1.2$ centimeters

$$C = 2\pi r \qquad r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\,dC = \frac{1}{2\pi}(56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $\dfrac{dA}{A} = \dfrac{(1/2\pi)C\,dC}{(1/4\pi)C^2} = \dfrac{2dC}{C} \quad 0.03$

$\dfrac{dC}{C} \quad \dfrac{0.03}{2} = 0.015 = 1.5\%$

36. $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, x changes from 115 to 120

$dP = (500 - 2x - x + 77)dx = (577 - 3x)\,dx = [577 - 3(115)](120 - 115) = 1160$

Approximate percentage change: $\frac{dP}{P}(100) = \frac{1160}{43517.50}(100) \approx 2.7\%$

38. $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 4\pi r^2\,dr = 4\pi(100)^2(0.2) = 8000\pi$ cm^3

40. $E = IR$

$$R = \frac{E}{I}$$

$$dR = -\frac{E}{I^2}dI$$

$$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

42. See Exercise 41.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5\cot\theta)(9.5) = 45.125\cot\theta$$

$$dA = -45.125\csc^2\theta\,d\theta$$

$$\left|\frac{dA}{A}\right| = \frac{\csc^2\theta\,d\theta}{\cot\theta} = \frac{d\theta}{\sin\theta\cos\theta}$$

$$= \frac{0.25°}{(\sin 26.75°)(\cos 26.75°)}$$

$$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$$

$$\approx 0.0109 = 1.09\% \text{ (in radians)}$$

44. $h = 50\tan\theta$

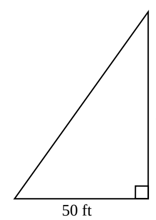
$\theta = 71.5° = 1.2479$ radians

$dh = 50\sec^2\theta \cdot d\theta$

$$\left|\frac{dh}{x}\right| = \left|\frac{50\sec^2(1.2479)}{50\tan(1.2479)}d\theta\right| \quad 0.06$$

$$\left|\frac{9.9316}{2.9886}d\theta\right| \quad 0.06$$

$$|d\theta| \quad 0.018$$



46. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$

$$f(x + \Delta x) \approx f(x) + f'(x)dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$

$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

48. Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2\,dx$$

$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

50. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

52. Propagated error $= f(x + \Delta x) - f(x)$,

relative error $= \left|\dfrac{dy}{y}\right|$, and the percent error $= \left|\dfrac{dy}{y}\right| \times 100$.

54. True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

56. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and

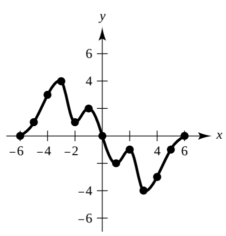
$$dy = f'(x)\,dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

Thus, $dy > \Delta y$ in this example.

Review Exercises for Chapter 3

2. (a) $f(4) = -f(-4) = -3$

(c)



At least six critical numbers on $(-6, 6)$.

(b) $f(-3) = -f(3) = -(-4) = 4$

(d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

(e) No, $\lim_{x \quad 0} f(x)$ exists because f is continuous at $(0, 0)$.

(f) Yes, f is differentiable at $x = 2$.

4. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$, $[0, 2]$

$$f'(x) = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $\left(2, 2/\sqrt{5}\right)$ Maximum

6. No. f is not differentiable at $x = 2$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

10. $f(x) = \dfrac{1}{x}$, $1 \quad x \quad 4$

$$f'(x) = -\frac{1}{x^2}$$

$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$

$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$

$$c = 2$$

12. $f(x) = \sqrt{x} - 2x$, $0 \quad x \quad 4$

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$

$$c = 1$$

14.
$$f(x) = 2x^2 - 3x + 1$$
$$f'(x) = 4x - 3$$
$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$
$$f'(c) = 4c - 3 = 5$$
$$c = 2 = \text{Midpoint of } [0, 4]$$

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

18. $f(x) = \sin x + \cos x,\ 0 \quad x \quad 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

20. $g(x) = \frac{3}{2}\sin\left(\frac{\pi x}{2} - 1\right),\ [0, 4]$

$$g'(x) = \frac{3}{2}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi x}{2} - 1\right)$$
$$= 0 \text{ when } x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$$

Relative maximum: $\left(1 + \frac{2}{\pi}, \frac{3}{2}\right)$

Relative minimum: $\left(3 + \frac{2}{\pi}, -\frac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

22. (a) $y = A\sin\left(\sqrt{k/m}\,t\right) + B\cos\left(\sqrt{k/m}\,t\right)$

$$y' = A\sqrt{k/m}\cos\left(\sqrt{k/m}\,t\right) - B\sqrt{k/m}\sin\left(\sqrt{k/m}\,t\right)$$
$$= 0 \text{ when } \frac{\sin\sqrt{k/m}\,t}{\cos\sqrt{k/m}\,t} = \frac{A}{B} \qquad \tan\left(\sqrt{k/m}\,t\right) = \frac{A}{B}.$$

Therefore,

$$\sin\left(\sqrt{k/m}\,t\right) = \frac{A}{\sqrt{A^2 + B^2}}$$
$$\cos\left(\sqrt{k/m}\,t\right) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi}\sqrt{k/m}$

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x = 0$ when $x = 0$.

Point of inflection: $(0, -16)$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

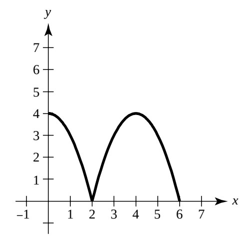
26. $h(t) = t - 4\sqrt{t + 1}$ Domain: $[-1, \infty)$

$h'(t) = 1 - \dfrac{2}{\sqrt{t + 1}} = 0 \qquad t = 3$

$h''(t) = \dfrac{1}{(t + 1)^{3/2}}$

$h''(3) = \dfrac{1}{8} > 0$ $(3, -5)$ is a relative minimum.

28.



30. $C = \left(\dfrac{Q}{x}\right)s + \left(\dfrac{x}{2}\right)r$

$\dfrac{dC}{dx} = -\dfrac{Qs}{x^2} + \dfrac{r}{2} = 0$

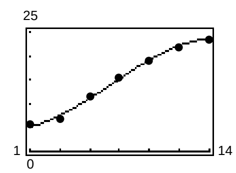
$\dfrac{Qs}{x^2} = \dfrac{r}{2}$

$x^2 = \dfrac{2Qs}{r}$

$x = \sqrt{\dfrac{2Qs}{r}}$

32. (a) $S = -0.1222t^3 + 1.3655t^2 - 0.9052t + 4.8429$

(b)



(c) $S'(t) = 0$ when $t = 3.7$. This is a maximum by the First Derivative Test.

(d) No, because the t^3 coefficient term is negative.

34. $\displaystyle\lim_{x \to \infty} \frac{2x}{3x^2 + 5} = \lim_{x \to \infty} \frac{2/x}{3 + 5/x^2} = 0$

36. $\displaystyle\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 4}} = \lim_{x \to \infty} \frac{3}{\sqrt{1 + 4/x^2}} = 3$

38. $g(x) = \dfrac{5x^2}{x^2 + 2}$

$\displaystyle\lim_{x \to \infty} \frac{5x^2}{x^2 + 2} = \lim_{x \to \infty} \frac{5}{1 + (2/x^2)} = 5$

Horizontal asymptote: $y = 5$

40. $f(x) = \dfrac{3x}{\sqrt{x^2 + 2}}$

$$\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \to \infty} \frac{3x/x}{\sqrt{x^2 + 2}/\sqrt{x^2}}$$

$$= \lim_{x \to \infty} \frac{3}{\sqrt{1 + (2/x^2)}} = 3$$

$$\lim_{x \to -\infty} \frac{3x}{\sqrt{x^2 + 2}} = \lim_{x \to -\infty} \frac{3x/x}{\sqrt{x^2 + 2}/\left(-\sqrt{x^2}\right)}$$

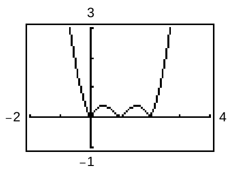
$$= \lim_{x \to -\infty} \frac{3}{-\sqrt{1 + (2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

42. $f(x) = |x^3 - 3x^2 + 2x| = |x(x-1)(x-2)|$

Relative minima: $(0, 0), (1, 0), (2, 0)$

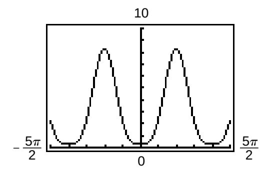
Relative maxima: $(1.577, 0.38), (0.423, 0.38)$



44. $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

Relative minima: $(2\pi k, 0.29)$ where k is any integer.

Relative maxima: $((2k-1)\pi, 8.29)$ where k is any integer.



46. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$ when $x = 0, 3$.

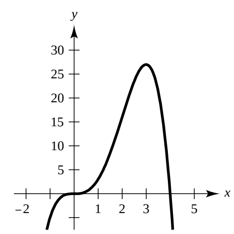
$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0$ when $x = 0, 2$.

$f''(3) < 0$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0), (2, 16)$

Intercepts: $(0, 0), (4, 0)$



48. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ is a relative maximum.

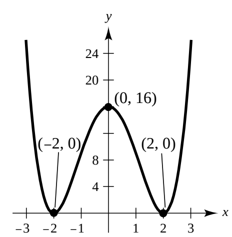
$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to y-axis



50. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16{,}875}{256}, \infty\right)$

$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$

$= (4x - 7)(x + 2)^2 = 0$ when $x = -2, \frac{7}{4}$.

$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$

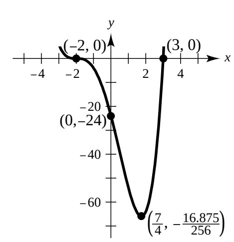
$= 6(2x - 1)(x + 2) = 0$ when $x = -2, \frac{1}{2}$.

$f''\left(\frac{7}{4}\right) > 0$

Therefore, $\left(\frac{7}{4}, -\frac{16{,}875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24), (3, 0)$



52. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

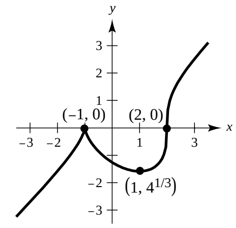
Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

$(-1, 0)$ is a relative maximum.

$\left(1, -\sqrt[3]{4}\right)$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0)$



54. $f(x) = \dfrac{2x}{1 + x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$$f'(x) = \frac{2(1 - x)(1 + x)}{(1 + x^2)^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = \frac{-2x(3 - x^2)}{(1 + x^2)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

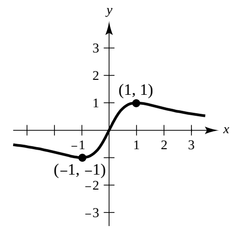
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $\left(-\sqrt{3}, -\sqrt{3}/2\right), (0, 0), \left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



56. $f(x) = \dfrac{x^2}{1 + x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \dfrac{1}{2}\right]$

$$f'(x) = \frac{(1 + x^4)(2x) - x^2(4x^3)}{(1 + x^4)^2} = \frac{2x(1 - x)(1 + x)(1 + x^2)}{(1 + x^4)^2} = 0 \text{ when } x = 0, \pm 1.$$

$$f''(x) = \frac{(1 + x^4)^2(2 - 10x^4) - (2x - 2x^5)(2)(1 + x^4)(4x^3)}{(1 + x^4)^4} = \frac{2(1 - 12x^4 + 3x^8)}{(1 + x^4)^3} = 0 \text{ when } x = \pm\sqrt[4]{\frac{6 \pm \sqrt{33}}{3}}.$$

$f''(\pm 1) < 0$

Therefore, $\left(\pm 1, \dfrac{1}{2}\right)$ are relative maxima.

$f''(0) > 0$

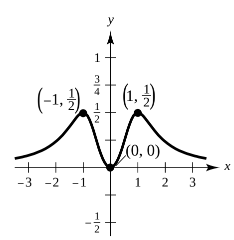
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm\sqrt[4]{\dfrac{6 - \sqrt{33}}{3}}, 0.29\right), \left(\pm\sqrt[4]{\dfrac{6 + \sqrt{33}}{3}}, 0.40\right)$

Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



58. $f(x) = x^2 + \dfrac{1}{x} = \dfrac{x^3 + 1}{x}$

Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = 2x - \dfrac{1}{x^2} = \dfrac{2x^3 - 1}{x^2} = 0$ when $x = \dfrac{1}{\sqrt[3]{2}}$.

$f''(x) = 2 + \dfrac{2}{x^3} = \dfrac{2(x^3 + 1)}{x^3} = 0$ when $x = -1$.

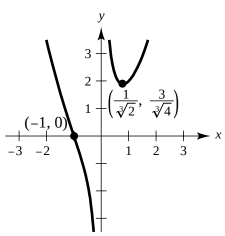
$f''\left(\dfrac{1}{\sqrt[3]{2}}\right) > 0$

Therefore, $\left(\dfrac{1}{\sqrt[3]{2}}, \dfrac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$

Intercept: $(-1, 0)$

Vertical asymptote: $x = 0$

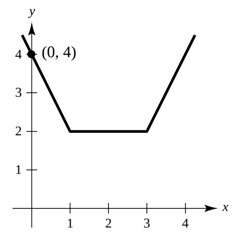


60. $f(x) = |x - 1| + |x - 3| = \begin{cases} -2x + 4, & x \quad 1 \\ 2, & 1 < x \quad 3 \\ 2x - 4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



62. $f(x) = \dfrac{1}{\pi}(2 \sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\dfrac{-3\sqrt{3}}{2\pi}, \dfrac{3\sqrt{3}}{2\pi}\right]$

$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2 \cos \pi x + 1)(\cos \pi x - 1) = 0$

Critical Numbers: $x = \pm\dfrac{2}{3}, 0$

$f''(x) = 2\pi(-\sin \pi x + 2 \sin 2\pi x) = 2\pi \sin \pi x(-1 + 4 \cos \pi x) = 0$ when $x = 0, \pm 1, \pm 0.420$.

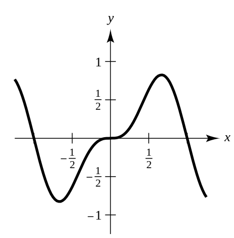
By the First Derivative Test: $\left(-\dfrac{2}{3}, \dfrac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\dfrac{2}{3}, \dfrac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462), (0.420, 0.462), (\pm 1, 0), (0, 0)$

Intercepts: $(-1, 0), (0, 0), (1, 0)$

Symmetric with respect to the origin



64. $f(x) = x^n$, n is a positive integer.

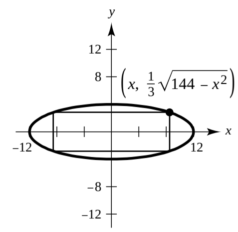
(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

(b) $f''(x) = n(n - 1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \quad 3$.

66. Ellipse: $\dfrac{x^2}{144} + \dfrac{y^2}{16} = 1,\ y = \dfrac{1}{3}\sqrt{144 - x^2}$



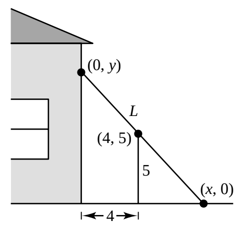
$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$

The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \dfrac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

68. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,



$$m = \frac{y - 5}{0 - 4} = \frac{5 - 0}{4 - x} \text{ or } y = \frac{5x}{x - 4}.$$

Let $f(x) = L^2 = x^2 + \left(\dfrac{5x}{x - 4}\right)^2$

$$f'(x) = 2x + 50\left(\frac{x}{x - 4}\right)\left[\frac{x - 4 - x}{(x - 4)^2}\right] = 0$$

$$x - \frac{100x}{(x - 4)^3} = 0$$

$$x[(x - 4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

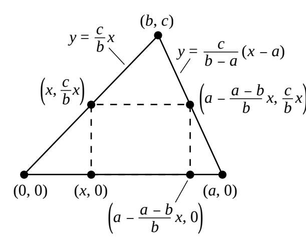
$$L = \sqrt{x^2 + \frac{25x^2}{(x - 4)^2}} = \frac{x}{x - 4}\sqrt{(x - 4)^2 + 25} = \frac{\sqrt[3]{100} + 4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$

70. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c). The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b - a}(\bar{x} - a)$$

$$(b - a)x = b(\bar{x} - a)$$

$$\bar{x} = \frac{b - a}{b}x + a = a - \frac{a - b}{b}x.$$



Finally, the lower right vertex is

$$\left(a - \frac{a - b}{b}x, 0\right).$$

Width of rectangle: $a - \dfrac{a - b}{b}x - x$

Height of rectangle: $\dfrac{c}{b}x$ (see figure)

$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a - b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

72. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y-b}{0-a} = \frac{b-0}{a-x} \qquad y = \frac{-bx}{a-x}.$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\frac{-bx}{a-x}\right)^2$.

$$f'(x) = 2x + 2\left(\frac{-bx}{a-x}\right)\left[\frac{-ab}{(a-x)^2}\right]$$

$$\frac{2x[(a-x)^3 + ab^2]}{(a-x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$L = \sqrt{\left(a + \sqrt[3]{ab^2}\right)^2 + \left(b + \sqrt[3]{a^2b}\right)^2}$$

$$= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2}$$

$$= (a^{2/3} + b^{2/3})^{3/2} \text{ meters}$$

74. Using Exercise 73 as a guide we have $L_1 = a \csc\theta$ and $L_2 = b \sec\theta$. Then $dL/d\theta = -a \csc\theta \cot\theta + b \sec\theta \tan\theta = 0$ when

$$\tan\theta = \sqrt[3]{a/b},\ \sec\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}},\ \csc\theta = \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} \text{ and}$$

$$L = L_1 + L_2 = a\csc\theta + b\sec\theta = a\frac{(a^{2/3} + b^{2/3})^{1/2}}{a^{1/3}} + b\frac{(a^{2/3} + b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

76. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500} + 7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\frac{dT}{dv} = \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41{,}250}{50v^2}$$

$$= 0 \text{ when } v = \sqrt{3750} = 25\sqrt{6} \approx 61.2 \text{ mph.}$$

$$\frac{d^2T}{dv^2} = \frac{1650}{v^3} > 0 \text{ when } v = 25\sqrt{6} \text{ so this value yields a minimum.}$$

78. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$f'(x) = 3x^2 + 2$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.