CHAPTER 4
Integration

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Section 4.1 Antiderivatives and Indefinite Integration

Solutions to Even-Numbered Exercises

2. $\frac{d}{dx}\left(x^4 + \frac{1}{x} + C\right) = 4x^3 - \frac{1}{x^2}$

4. $\frac{d}{dx}\left(\frac{2(x^2+3)}{3\sqrt{x}} + C\right) = \frac{d}{dx}\left(\frac{2}{3}x^{3/2} + 2x^{-1/2} + C\right)$

$= x^{1/2} - x^{-3/2} = \frac{x^2 - 1}{x^{3/2}}$

6. $\frac{dr}{d\theta} = \pi$

$r = \pi\theta + C$

Check: $\frac{d}{d\theta}[\pi\theta + C] = \pi$

8. $\frac{dy}{dx} = 2x^{-3}$

$y = \frac{2x^{-2}}{-2} + C = \frac{-1}{x^2} + c$

Check: $\frac{d}{dx}\left[\frac{-1}{x^2} + C\right] = 2x^{-3}$

	Given	*Rewrite*	*Integrate*	*Simplify*
10.	$\int \frac{1}{x^2}\,dx$	$\int x^{-2}\,dx$	$\frac{x^{-1}}{-1} + C$	$-\frac{1}{x} + C$
12.	$\int x(x^2+3)\,dx$	$\int (x^3 + 3x)\,dx$	$\frac{x^4}{4} + 3\left(\frac{x^2}{2}\right) + C$	$\frac{1}{4}x^4 + \frac{3}{2}x^2 + C$
14.	$\int \frac{1}{(3x^2)}dx$	$\frac{1}{9}\int x^{-2}dx$	$\frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{-1}{9x} + C$

16. $\int (5 - x)dx = 5x - \frac{x^2}{2} + C$

Check: $\frac{d}{dx}\left[5x - \frac{x^2}{2} + C\right] = 5 - x$

18. $\int (4x^3 + 6x^2 - 1)dx = x^4 + 2x^3 - x + C$

Check: $\frac{d}{dx}[x^4 + 2x^3 - x + C] = 4x^3 + 6x^2 - 1$

20. $\int (x^3 - 4x + 2)dx = \frac{x^4}{4} - 2x^2 + 2x + C$

Check: $\frac{d}{dx}\left[\frac{x^4}{4} - 2x^2 + 2x + C\right] = x^3 - 4x + 2$

22. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int \left(x^{1/2} + \frac{1}{2}x^{-1/2}\right) dx = \frac{x^{3/2}}{3/2} + \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \frac{2}{3}x^{3/2} + x^{1/2} + C$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + x^{1/2} + C\right) = x^{1/2} + \frac{1}{2}x^{-1/2} = \sqrt{x} + \frac{1}{2\sqrt{x}}$

24. $\int\left(\sqrt[4]{x^3} + 1\right) dx = \int (x^{3/4} + 1)\, dx = \frac{4}{7}x^{7/4} + x + C$

Check: $\frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1$

26. $\int \frac{1}{x^4}\, dx = \int x^{-4}\, dx = \frac{x^{-3}}{-3} + C = -\frac{1}{3x^3} + C$

Check: $\frac{d}{dx}\left(-\frac{1}{3x^3} + C\right) = \frac{1}{x^4}$

28. $\int \frac{x^2 + 2x - 3}{x^4} dx = \int (x^{-2} + 2x^{-3} - 3x^{-4}) dx$

$$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C$$

$$= \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C$$

Check: $\frac{d}{dx}\left[-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C\right] = x^{-2} + 2x^{-3} - 3x^{-4}$

$$= \frac{x^2 + 2x - 3}{x^4}$$

30. $\int (2t^2 - 1)^2\, dt = \int (4t^4 - 4t^2 + 1)\, dt$

$$= \frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C$$

Check: $\frac{d}{dt}\left(\frac{4}{5}t^5 - \frac{4}{3}t^3 + t + C\right) = 4t^4 - 4t^2 + 1$

$$= (2t^2 - 1)^2$$

32. $\int (1 + 3t)t^2\, dt = \int (t^2 + 3t^3)\, dt = \frac{1}{3}t^3 + \frac{3}{4}t^4 + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 = (1 + 3t)t^2$

34. $\int 3\, dt = 3t + C$

Check: $\frac{d}{dt}(3t + C) = 3$

36. $\int (t^2 - \sin t)\, dt = \frac{1}{3}t^3 + \cos t + C$

Check: $\frac{d}{dt}\left(\frac{1}{3}t^3 + \cos t + C\right) = t^2 - \sin t$

38. $\int (\theta^2 + \sec^2 \theta)\, d\theta = \frac{1}{3}\theta^3 + \tan \theta + C$

Check: $\frac{d}{d\theta}\left(\frac{1}{3}\theta^3 + \tan \theta + C\right) = \theta^2 + \sec^2 \theta$

40. $\int \sec y(\tan y - \sec y)\, dy = \int (\sec y \tan y - \sec^2 y)\, dy$

$$= \sec y - \tan y + C$$

Check: $\frac{d}{dy}(\sec y - \tan y + C) = \sec y \tan y - \sec^2 y$

$$= \sec y(\tan y - \sec y)$$

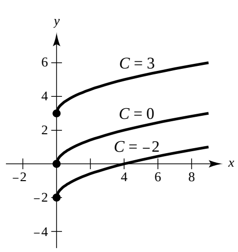
42. $\int \frac{\cos x}{1 - \cos^2 x}\, dx = \int \frac{\cos x}{\sin^2 x}\, dx = \int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) dx$

$$= \int \csc x \cot x\, dx = -\csc x + C$$

Check: $\frac{d}{dx}[-\csc x + C] = \csc x \cot x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$

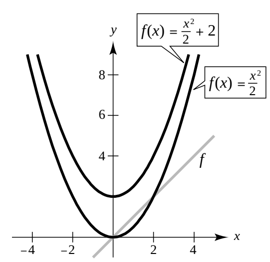
$$= \frac{\cos x}{1 - \cos^2 x}$$

44. $f(x) = \sqrt{x}$



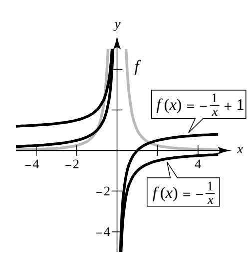
46. $f'(x) = x$

$$f(x) = \frac{x^2}{2} + C$$



48. $f'(x) = \frac{1}{x^2}$

$$f(x) = -\frac{1}{x} + C$$



50. $\frac{dy}{dx} = 2(x-1) = 2x-2,\ (3, 2)$

$y = \int 2(x-1)\,dx = x^2 - 2x + C$

$2 = (3)^2 - 2(3) + C \qquad C = -1$

$y = x^2 - 2x - 1$

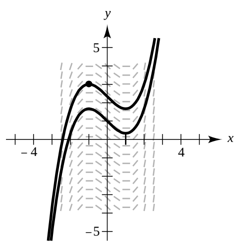
52. $\frac{dy}{dx} = -\frac{1}{x^2} = -x^{-2},\ (1,3)$

$y = \int -x^{-2}\,dx = \frac{1}{x} + C$

$3 = \frac{1}{1} + C \qquad C = 2$

$y = \frac{1}{x} + 2,\ x > 0$

54. (a)



(b) $\frac{dy}{dx} = x^2 - 1,\ (-1, 3)$

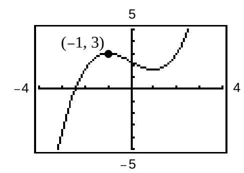
$y = \frac{x^3}{3} - x + C$

$3 = \frac{(-1)^3}{3} - (-1) + C$

$3 = -\frac{1}{3} + 1 + C$

$C = \frac{7}{3}$

$y = \frac{x^3}{3} - x + \frac{7}{3}$



56. $g'(x) = 6x^2, g(0) = -1$

$g(x) = \int 6x^2\,dx = 2x^3 + C$

$g(0) = -1 = 2(0)^3 + C \qquad C = -1$

$g(x) = 2x^3 - 1$

58. $f'(s) = 6s - 8s^3,\ f(2) = 3$

$f(s) = \int (6s - 8s^3)ds = 3s^2 - 2s^4 + C$

$f(2) = 3 = 3(2)^2 - 2(2)^4 + C = 12 - 32 + C \qquad C = 23$

$f(s) = 3s^2 - 2s^4 + 23$

60. $f''(x) = x^2$

$f'(0) = 6$

$f(0) = 3$

$f'(x) = \int x^2\,dx = \frac{1}{3}x^3 + C_1$

$f'(0) = 0 + C_1 = 6 \qquad C_1 = 6$

$f'(x) = \frac{1}{3}x^3 + 6$

$f(x) = \int \left(\frac{1}{3}x^3 + 6\right)dx = \frac{1}{12}x^4 + 6x + C_2$

$f(0) = 0 + 0 + C_2 = 3 \qquad C_2 = 3$

$f(x) = \frac{1}{12}x^4 + 6x + 3$

62. $f''(x) = \sin x$

$f'(0) = 1$

$f(0) = 6$

$f'(x) = \int \sin x\,dx = -\cos x + C_1$

$f'(0) = -1 + C_1 = 1 \qquad C_1 = 2$

$f'(x) = -\cos x + 2$

$f(x) = \int (-\cos x + 2)\,dx = -\sin x + 2x + C_2$

$f(0) = 0 + 0 + C_2 = 6 \qquad C_2 = 6$

$f(x) = -\sin x + 2x + 6$

64. $\dfrac{dP}{dt} = k\sqrt{t}, \; 0 \quad t \quad 10$

$$P(t) = \int kt^{1/2}\,dt = \frac{2}{3}kt^{3/2} + C$$

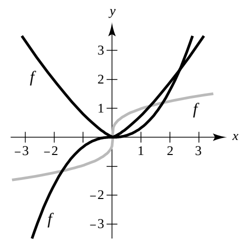
$$P(0) = 0 + C = 500 \qquad C = 500$$

$$P(1) = \frac{2}{3}k + 500 = 600 \qquad k = 150$$

$$P(t) = \frac{2}{3}(150)t^{3/2} + 500 = 100t^{3/2} + 500$$

$$P(7) = 100(7)^{3/2} + 500 \approx 2352 \text{ bacteria}$$

66. Since f'' is negative on $(-\infty, 0)$, f' is decreasing on $(-\infty, 0)$. Since f'' is positive on $(0, \infty)$, f' is increasing on $(0, \infty)$. f' has a relative minimum at $(0, 0)$. Since f' is positive on $(-\infty, \infty)$, f is increasing on $(-\infty, \infty)$.



68. $f''(t) = a(t) = -32 \text{ ft/sec}^2$

$$f'(0) = v_0$$

$$f(0) = s_0$$

$$f'(t) = v(t) = \int -32\,dt = -32t + C_1$$

$$f'(0) = 0 + C_1 = v_0 \qquad C_1 = v_0$$

$$f'(t) = -32t + v_0$$

$$f(t) = s(t) = \int (-32t + v_0)\,dt = -16t^2 + v_0t + C_2$$

$$f(0) = 0 + 0 + C_2 = s_0 \qquad C_2 = s_0$$

$$f(t) = -16t^2 + v_0t + s_0$$

70. $v_0 = 16$ ft/sec

$s_0 = 64$ ft

(a) $$s(t) = -16t^2 + 16t + 64 = 0$$

$$-16(t^2 - t - 4) = 0$$

$$t = \frac{1 \pm \sqrt{17}}{2}$$

Choosing the positive value,

$$t = \frac{1 + \sqrt{17}}{2} \approx 2.562 \text{ seconds.}$$

(b) $$v(t) = s'(t) = -32t + 16$$

$$v\left(\frac{1 + \sqrt{17}}{2}\right) = -32\left(\frac{1 + \sqrt{17}}{2}\right) + 16$$

$$= -16\sqrt{17} \approx -65.970 \text{ ft/sec}$$

72. From Exercise 71, $f(t) = -4.9t^2 + 1600$. (Using the canyon floor as position 0.)

$$f(t) = 0 = -4.9t^2 + 1600$$

$$4.9t^2 = 1600$$

$$t^2 = \frac{1600}{4.9} \qquad t \approx \sqrt{326.53} \approx 18.1 \text{ sec}$$

74. From Exercise 71, $f(t) = -4.9t^2 + v_0t + 2$. If

$$f(t) = 200 = -4.9t^2 + v_0t + 2,$$

then

$$v(t) = -9.8t + v_0 = 0$$

for this t value. Hence, $t = v_0/9.8$ and we solve

$$-4.9\left(\frac{v_0}{9.8}\right)^2 + v_0\left(\frac{v_0}{9.8}\right) + 2 = 200$$

$$\frac{-4.9\,v_0^2}{(9.8)^2} + \frac{v_0^2}{9.8} = 198$$

$$-4.9\,v_0^2 + 9.8\,v_0^2 = (9.8)^2\,198$$

$$4.9\,v_0^2 = (9.8)^2\,198$$

$$v_0^2 = 3880.8 \qquad v_0 \approx 62.3 \text{ m/sec.}$$

76. $\int v\,dv = -GM\int \frac{1}{y^2}\,dy$

$$\frac{1}{2}v^2 = \frac{GM}{y} + C$$

When $y = R$, $v = v_0$.

$$\frac{1}{2}v_0^{\,2} = \frac{GM}{R} + C$$

$$C = \frac{1}{2}v_0^{\,2} - \frac{GM}{R}$$

$$\frac{1}{2}v^2 = \frac{GM}{y} + \frac{1}{2}v_0^{\,2} - \frac{GM}{R}$$

$$v^2 = \frac{2GM}{y} + v_0^{\,2} - \frac{2GM}{R}$$

$$v^2 = v_0^{\,2} + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

78. $x(t) = (t-1)(t-3)^2 \qquad 0 \quad t \quad 5$

$= t^3 - 7t^2 + 15t - 9$

(a) $v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3)$

$a(t) = v'(t) = 6t - 14$

(b) $v(t) > 0$ when $0 < t < \frac{5}{3}$ and $3 < t < 5$.

(c) $a(t) = 6t - 14 = 0$ when $t = \frac{7}{3}$.

$$v\left(\frac{7}{3}\right) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3}$$

80. (a) $a(t) = \cos t$

$$v(t) = \int a(t)\,dt = \int \cos t\,dt = \sin t + C_1 = \sin t \ \text{ (since } v_0 = 0)$$

$$f(t) = \int v(t)\,dt = \int \sin t\,dt = -\cos t + C_2$$

$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \qquad C_2 = 4$$

$$f(t) = -\cos t + 4$$

(b) $v(t) = 0 = \sin t$ for $t = k\pi$, $k = 0, 1, 2, \ldots$

82. $v(0) = 45 \text{ mph} = 66 \text{ ft/sec}$

$30 \text{ mph} = 44 \text{ ft/sec}$

$15 \text{ mph} = 22 \text{ ft/sec}$

$a(t) = -a$

$v(t) = -at + 66$

$s(t) = -\frac{a}{2}t^2 + 66t$ (Let $s(0) = 0$.)

$v(t) = 0$ after car moves 132 ft.

$-at + 66 = 0$ when $t = \frac{66}{a}$.

$$s\left(\frac{66}{a}\right) = -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right)$$

$$= 132 \text{ when } a = \frac{33}{2} = 16.5.$$

$a(t) = -16.5$

$v(t) = -16.5t + 66$

$s(t) = -8.25t^2 + 66t$

(a) $-16.5t + 66 = 44$

$$t = \frac{22}{16.5} \approx 1.333$$

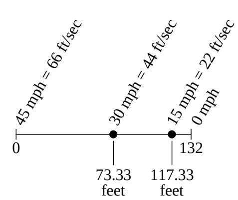
$$s\left(\frac{22}{16.5}\right) \approx 73.33 \text{ ft}$$

(b) $-16.5t + 66 = 22$

$$t = \frac{44}{16.5} \approx 2.667$$

$$s\left(\frac{44}{16.5}\right) \approx 117.33 \text{ ft}$$

(c)



It takes 1.333 seconds to reduce the speed from 45 mph to 30 mph, 1.333 seconds to reduce the speed from 30 mph to 15 mph, and 1.333 seconds to reduce the speed from 15 mph to 0 mph. Each time, less distance is needed to reach the next speed reduction.

84. No, car 2 will be ahead of car 1. If $v_1(t)$ and $v_2(t)$ are the respective velocities, then $\int_0^{30} |v_2(t)|\,dt > \int_0^{30} |v_1(t)|\,dt$.

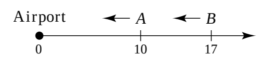
86. (a) $v = 0.6139t^3 - 5.525t^2 + 0.0492t + 65.9881$

(b) $s(t) = \int v(t)dt = \dfrac{0.6139t^4}{4} - \dfrac{5.525t^3}{3} + \dfrac{0.0492t^2}{2} + 65.9881t$

(**Note:** Assume $s(0) = 0$ is initial position)

$s(6) \approx 196.1$ feet

88. Let the aircrafts be located 10 and 17 miles away from the airport, as indicated in the figure.



$v_A(t) = k_A t - 150$ $\qquad v_B = k_B t - 250$

$s_A(t) = \frac{1}{2}k_A t^2 - 150t + 10$ $\qquad s_B = \frac{1}{2}k_B t^2 - 250t + 17$

(a) When aircraft A lands at time t_A you have

$$v_A(t_A) = k_A t_A - 150 = -100 \qquad k_A = \frac{50}{t_A}$$

$$s_A(t_A) = \frac{1}{2}k_A t_A^2 - 150t_A + 10 = 0$$

$$\frac{1}{2}\left(\frac{50}{t_A}\right)t_A^2 - 150t_A = -10$$

$$125t_A = 10$$

$$t_A = \frac{10}{125}.$$

$$k_A = \frac{50}{t_A} = 50\left(\frac{125}{10}\right) = 625 \qquad S_A(t) = S_1(t) = \frac{625}{2}t^2 - 150t + 10$$

Similarly, when aircraft B lands at time t_B you have

$$v_B(t_B) = k_B t_B - 250 = -115 \qquad k_B = \frac{135}{t_B}$$

$$s_B(t_B) = \frac{1}{2}k_B t_B^2 - 250t_B + 17 = 0$$

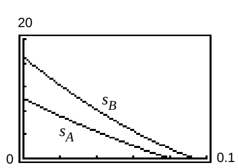
$$\frac{1}{2}\left(\frac{135}{t_B}\right)t_B^2 - 250t_B = -17$$

$$\frac{365}{2}t_B = 17$$

$$t_B = \frac{34}{365}.$$

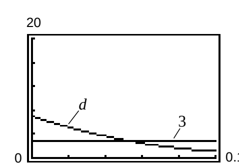
$$k_B = \frac{135}{t_B} = 135\left(\frac{365}{34}\right) = \frac{49{,}275}{34} \qquad S_B(t) = S_2(t) = \frac{49{,}275}{68}t^2 - 250t + 17$$

(b)



(c) $d = s_B(t) - s_A(t)$

Yes, $d < 3$ for $t > 0.0505$.



90. True

92. True

94. False. f has an infinite number of antiderivatives, each differing by a constant.

96. $\frac{d}{dx}\left[[s(x)]^2 + [c(x)]^2\right] = 2s(x)s'(x) + 2c(x)c'(x)$

$= 2s(x)c(x) - 2c(x)s(x)$

$= 0$

Thus, $[s(x)]^2 + [c(x)]^2 = k$ for some constant k. Since,

$s(0) = 0$ and $c(0) = 1$, $k = 1$.

Therefore,

$[s(x)]^2 + [c(x)]^2 = 1.$

[Note that $s(x) = \sin x$ and $c(x) = \cos x$ satisfy these properties.]

Section 4.2 Area

2. $\sum_{k=3}^{6} k(k-2) = 3(1) + 4(2) + 5(3) + 6(4) = 50$

4. $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

6. $\sum_{i=1}^{4} [(i-1)^2 + (i+1)^3] = (0+8) + (1+27) + (4+64) + (9+125) = 238$

8. $\sum_{i=1}^{15} \frac{5}{1+i}$

10. $\sum_{j=1}^{4} \left[1 - \left(\frac{j}{4}\right)^2\right]$

12. $\frac{2}{n}\sum_{i=1}^{n} \left[1 - \left(\frac{2i}{n} - 1\right)^2\right]$

14. $\frac{1}{n}\sum_{i=0}^{n-1} \sqrt{1 - \left(\frac{i}{n}\right)^2}$

16. $\sum_{i=1}^{15} (2i - 3) = 2\sum_{i=1}^{15} i - 3(15)$

$= 2\left[\frac{15(16)}{2}\right] - 45 = 195$

18. $\sum_{i=1}^{10} (i^2 - 1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1$

$= \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

20. $\sum_{i=1}^{10} i(i^2 + 1) = \sum_{i=1}^{10} i^3 + \sum_{i=1}^{10} i$

$= \frac{10^2(11)^2}{4} + \left[\frac{10(11)}{2}\right] = 3080$

22. sum seq(x ^ 3 − 2x, x, 1, 15, 1) = 14,160 (*TI-82*)

$\sum_{i=1}^{15} (i^3 - 2i) = \frac{(15)^2(15+1)^2}{4} - 2\,\frac{15(15+1)}{2}$

$= \frac{(15)^2(16)^2}{4} - 15(16) = 14{,}160$

24. $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

26. $S = \left[5 + 2 + 1 + \frac{2}{3} + \frac{1}{2}\right] = \frac{55}{6}$

$s = \left[2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right] = \frac{9}{2} = 4.5$

28. $S(8) = \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{1} + 2\right)\frac{1}{4}$

$+ \left(\sqrt{\frac{5}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{3}{2}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{2} + 2\right)\frac{1}{4}$

$= \frac{1}{4}\left(16 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{5}}{2} + \frac{\sqrt{6}}{2} + \frac{\sqrt{7}}{2} + \sqrt{2}\right) \approx 6.038$

$s(8) = (0 + 2)\frac{1}{4} + \left(\sqrt{\frac{1}{4}} + 2\right)\frac{1}{4} + \left(\sqrt{\frac{1}{2}} + 2\right)\frac{1}{4} + \cdots + \left(\sqrt{\frac{7}{4}} + 2\right)\frac{1}{4} \approx 5.685$

30. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

32. $\lim_{n \to \infty}\left[\left(\frac{64}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right] = \frac{64}{6}\lim_{n \to \infty}\left[\frac{2n^3 + 3n^2 + n}{n^3}\right] = \frac{64}{6}(2) = \frac{64}{3}$

34. $\lim_{n \to \infty}\left[\left(\frac{1}{n^2}\right)\frac{n(n+1)}{2}\right] = \frac{1}{2}\lim_{n \to \infty}\left[\frac{n^2 + n}{n^2}\right] = \frac{1}{2}(1) = \frac{1}{2}$

36. $\sum_{j=1}^{n}\frac{4j+3}{n^2} = \frac{1}{n^2}\sum_{j=1}^{n}(4j+3) = \frac{1}{n^2}\left[\frac{4n(n+1)}{2} + 3n\right] = \frac{2n+5}{n} = S(n)$

$S(10) = \frac{25}{10} = 2.5$

$S(100) = 2.05$

$S(1000) = 2.005$

$S(10{,}000) = 2.0005$

38. $\sum_{i=1}^{n}\frac{4i^2(i-1)}{n^4} = \frac{4}{n^4}\sum_{i=1}^{n}(i^3 - i^2) = \frac{4}{n^4}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{4}{n^3}\left[\frac{n^3 + 2n^2 + n}{4} - \frac{2n^2 + 3n + 1}{6}\right]$

$= \frac{1}{3n^3}[3n^3 + 6n^2 + 3n - 4n^2 - 6n - 2]$

$= \frac{1}{3n^3}[3n^3 + 2n^2 - 3n - 2] = S(n)$

$S(10) = 1.056$

$S(100) = 1.006566$

$S(1000) = 1.00066567$

$S(10{,}000) = 1.000066657$

40. $\lim_{n \to \infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n \to \infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n \to \infty}\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n \to \infty}\frac{4}{2}\left(1 + \frac{1}{n}\right) = 2$

42. $\lim_{n \to \infty}\sum_{i=1}^{n}\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n \to \infty}\frac{2}{n^3}\sum_{i=1}^{n}(n + 2i)^2$

$= \lim_{n \to \infty}\frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$

$= \lim_{n \to \infty}\frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$

$= 2\lim_{n \to \infty}\left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right]$

$= 2\left(1 + 2 + \frac{4}{3}\right) = \frac{26}{3}$

44. $\displaystyle \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^3 \left(\frac{2}{n}\right) = 2 \lim_{n \to \infty} \frac{1}{n^4} \sum_{i=1}^{n} (n + 2i)^3$

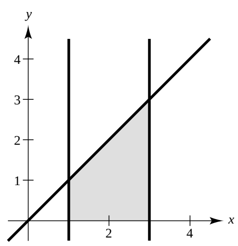
$$= 2 \lim_{n \to \infty} \frac{1}{n^4} \sum_{i=1}^{n} (n^3 + 6n^2 i + 12ni^2 + 8i^3)$$

$$= 2 \lim_{n \to \infty} \frac{1}{n^4} \left[n^4 + 6n^2 \left(\frac{n(n+1)}{2}\right) + 12n\left(\frac{n(n+1)(2n+1)}{6}\right) + 8\left(\frac{n^2(n+1)^2}{4}\right) \right]$$

$$= 2 \lim_{n \to \infty} \left(1 + 3 + \frac{3}{n} + 4 + \frac{6}{n} + \frac{2}{n^2} + 2 + \frac{4}{n} + \frac{2}{n^2}\right)$$

$$= 2 \lim_{n \to \infty} \left(10 + \frac{13}{n} + \frac{4}{n^2}\right) = 20$$

46. (a)



(b) $\Delta x = \dfrac{3 - 1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Since $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \sum_{i=1}^{n} f\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

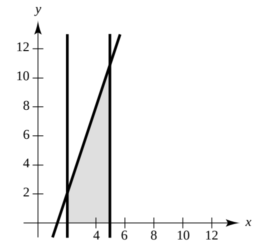
(f) $\displaystyle \lim_{n \to \infty} \sum_{i=1}^{n} \left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n \to \infty} \left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right]$

$$= \lim_{n \to \infty} \left[2 + \frac{2n+2}{n} - \frac{4}{n}\right] = \lim_{n \to \infty} \left[4 - \frac{2}{n}\right] = 4$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n \to \infty} \frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right]$$

$$= \lim_{n \to \infty} \left[2 + \frac{2(n+1)}{n}\right] = \lim_{n \to \infty} \left[4 + \frac{2}{n}\right] = 4$$

48. $y = 3x - 4$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \frac{5-2}{n} = \frac{3}{n}\right)$



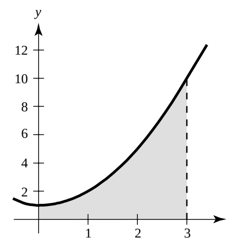
$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^{n} i - 12$$

$$= 6 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 6 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + \frac{27}{2} = \frac{39}{2}$$

50. $y = x^2 + 1$ on $[0, 3]$. $\left(\textbf{Note: } \Delta x = \frac{3-0}{n} = \frac{3}{n}\right)$



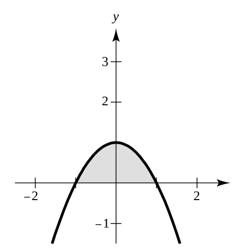
$$S(n) = \sum_{l=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{l=1}^{n}\left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right)$$

$$= \frac{27}{n^3}\sum_{l=1}^{n} l^2 + \frac{3}{n}\sum_{l=1}^{n} 1$$

$$= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}(n) = \frac{9}{2}\frac{2n^2 + 3n + 1}{n^2} + 3$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{9}{2}(2) + 3 = 12$$

52. $y = 1 - x^2$ on $[-1, 1]$. Find area of region over the interval $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1}{n}\right)$



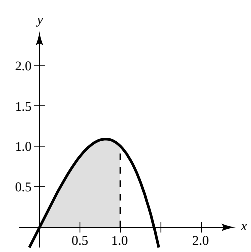
$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[1 - \left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 1 - \frac{n(n+1)(2n+1)}{6n^3} = 1 - \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\text{Area} = \frac{4}{3}$$

54. $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \frac{1-0}{n} = \frac{1}{n}\right)$



Since y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

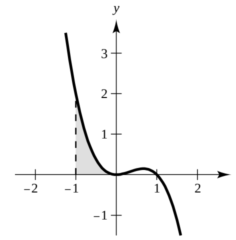
$$T(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$

56. $y = x^2 - x^3$ on $[-1, 0]$. $\left(\textbf{Note: } \Delta x = \dfrac{0 - (-1)}{n} = \dfrac{1}{n}\right)$



$$s(n) = \sum_{i=1}^{n} f\left(-1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{i}{n}\right)^2 - \left(-1 + \frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\left(2 - \frac{5i}{n} + \frac{4i^2}{n^2} - \frac{i^3}{n^3}\right)\left(\frac{1}{n}\right)\right] = 2 - \frac{5}{n^2}\sum_{i=1}^{n} i + \frac{4}{n^3}\sum_{i=1}^{n} i^2 - \frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= 2 - \frac{5}{2} - \frac{5}{2n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^3} - \frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2}$$

$$\text{Area} = \lim_{n \quad \infty} s(n) = 2 - \frac{5}{2} + \frac{4}{3} - \frac{1}{4} = \frac{7}{12}$$

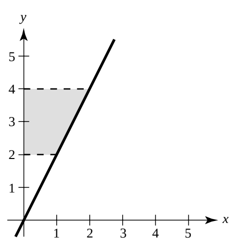
58. $g(y) = \frac{1}{2}y, 2 \quad y \quad 4.$ $\left(\textbf{Note: } \Delta y = \dfrac{4 - 2}{n} = \dfrac{2}{n}\right)$

$$S(n) = \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)$$

$$= \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n}$$

$$\text{Area} = \lim_{n \quad \infty} S(n) = 2 + 1 = 3$$



60. $f(y) = 4y - y^2, 1 \quad y \quad 2.$ $\left(\textbf{Note: } \Delta y = \dfrac{2 - 1}{n} = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\right]$$

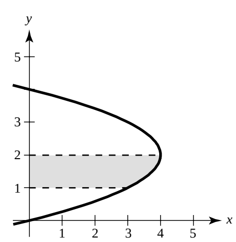
$$= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right)$$

$$= \frac{1}{n}\left[3n + \frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6}$$

$$\text{Area} = \lim_{n \quad \infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



62. $h(y) = y^3 + 1, 1 \quad y \quad 2$ $\left(\textbf{Note: } \Delta y = \dfrac{1}{n}\right)$

$$S(n) = \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right)$$

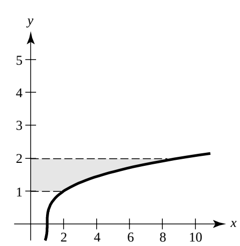
$$= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^3 + 1\right]\frac{1}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right)$$

$$= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{n(n+1)}{2}\right]$$

$$= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}$$

$$\text{Area} = \lim_{n \quad \infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$



64. $f(x) = x^2 + 4x,\ 0 \quad x \quad 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4} [c_i^2 + 4c_i](1)$$

$$= \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right]$$

$$= 53$$

66. $f(x) = \sin x,\ 0 \quad x \quad \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$$\Delta x = \frac{\pi}{8},\ c_1 = \frac{\pi}{16},\ c_2 = \frac{3\pi}{16},\ c_3 = \frac{5\pi}{16},\ c_4 = \frac{7\pi}{16}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4} (\sin c_i)\left(\frac{\pi}{8}\right)$$

$$= \frac{\pi}{8}\left(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}\right) \approx 1.006$$

68. $f(x) = \dfrac{8}{x^2 + 1}$ on $[2, 6]$.

n	4	8	12	16	20
Approximate area	2.3397	2.3755	2.3824	2.3848	2.3860

70. $f(x) = \cos\sqrt{x}$ on $[0, 2]$.

n	4	8	12	16	20
Approximate area	1.1041	1.1053	1.1055	1.1056	1.1056

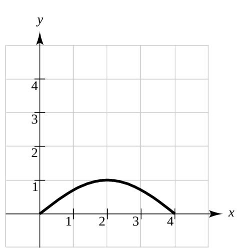
72. See the Definition of Area. Page 259.

74. $f(x) = \sqrt[3]{x},\ 0 \quad x \quad 8$

n	10	20	50	100	200
$s(n)$	10.998	11.519	11.816	11.910	11.956
$S(n)$	12.598	12.319	12.136	12.070	12.036
$M(n)$	12.040	12.016	12.005	12.002	12.001

(**Note:** exact answer is 12.)

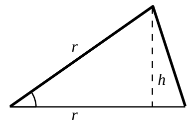
76.



a. $A \approx 3$ square units

78. True. (Theorem 4.3)

80. (a) $\theta = \dfrac{2\pi}{n}$



(b) $\sin\theta = \dfrac{h}{r}$

$$h = r\sin\theta$$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $A_n = n\left(\dfrac{1}{2}r^2\sin\dfrac{2\pi}{n}\right) = \dfrac{r^2n}{2}\sin\dfrac{2\pi}{n} = \pi r^2\left(\dfrac{\sin(2\pi/n)}{2\pi/n}\right)$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0}\pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

82. (a) $\displaystyle\sum_{i=1}^{n} 2i = n(n+1)$

The formula is true for $n = 1$: $2 = 1(1+1) = 2$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} 2i = k(k+1).$$

Then we have $\displaystyle\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1)$

$$= k(k+1) + 2(k+1)$$

$$= (k+1)(k+2)$$

Which shows that the formula is true for $n = k+1$.

(b) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

The formula is true for $n = 1$ because

$$1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Assume that the formula is true for $n = k$:

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Then we have $\displaystyle\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2}{4}(k+2)^2$$

which shows that the formula is true for $n = k+1$.

Section 4.3 Riemann Sums and Definite Integrals

2. $f(x) = \sqrt[3]{x}$, $y = 0$, $x = 0$, $x = 1$, $c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(3i^3 - 3i^2 + i)$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}$$

4. $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \frac{3-(-2)}{n} = \frac{5}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i$$

$$= -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

6. $y = 3x^2$ on $[1, 3]$. $\left(\text{Note: } \Delta x = \frac{3-1}{n} = \frac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} 3\left(1 + \frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$$

$$= \frac{6}{n}\sum_{i=1}^{n}\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \frac{6}{n}\left[n + \frac{4}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 6 + 12\frac{n+1}{n} + 4\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{1}^{3} 3x^2\,dx = \lim_{n\to\infty}\left[6 + \frac{12(n+1)}{n} + \frac{4(n+1)(2n+1)}{n^2}\right]$$

$$= 6 + 12 + 8 = 26$$

8. $y = 3x^2 + 2$ on $[-1, 2]$. $\left(\textbf{Note: } \Delta x = \frac{2-(-1)}{n} = \frac{3}{n}; \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(-1 + \frac{3i}{n}\right)^2 + 2\right]$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[3\left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 2\right]$$

$$= \frac{3}{n}\left[3n - \frac{18}{n}\frac{n(n+1)}{2} + \frac{27}{n^2}\frac{n(n+1)(2n+1)}{6} + 2n\right]$$

$$= 15 - \frac{27(n+1)}{n} + \frac{27}{2}\frac{(n+1)(2n+1)}{n^2}$$

$$\int_{-1}^{2}(3x^2+2)dx = \lim_{n\to\infty}\left[15 - 27\frac{(n+1)}{n} + \frac{27}{2}\frac{(n+1)(2n+1)}{n^2}\right]$$

$$= 15 - 27 + 27 = 15$$

10. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 6c_i(4-c_i)^2\,\Delta x_i = \int_0^4 6x(4-x)^2\,dx$

on the interval $[0, 4]$.

12. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\left(\frac{3}{c_i^2}\right)\Delta x_i = \int_1^3 \frac{3}{x^2}\,dx$

on the interval $[1, 3]$.

14. $\displaystyle\int_0^2 (4-2x)\,dx$

16. $\displaystyle\int_0^2 x^2\,dx$

18. $\displaystyle\int_{-1}^{1}\frac{1}{x^2+1}\,dx$

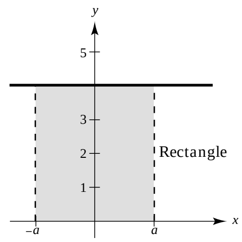
20. $\displaystyle\int_0^{\pi/4}\tan x\,dx$

22. $\int_0^2 (y-2)^2\,dy$

24. Rectangle

$A = bh = 2(4)(a)$

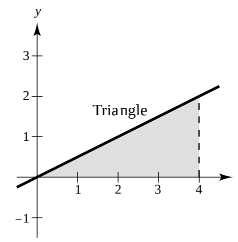
$A = \int_{-a}^{a} 4\,dx = 8a$



26. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(2)$

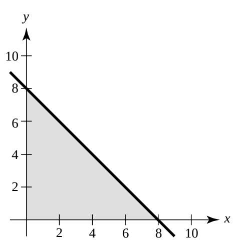
$A = \int_0^4 \frac{x}{2}\,dx = 4$



28. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(8) = 32$

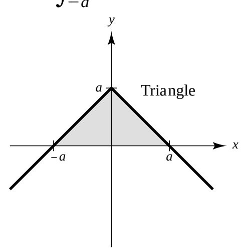
$A = \int_0^8 (8-x)dx = 32$



30. Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(2a)a$

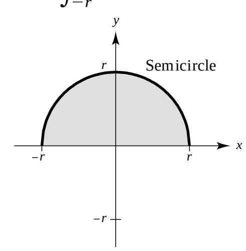
$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$



32. Semicircle

$A = \frac{1}{2}\pi r^2$

$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2$



In Exercises 34–40, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$.

34. $\int_2^2 x^3\,dx = 0$

36. $\int_2^4 15\,dx = 15\int_2^4 dx = 15(2) = 30$

38. $\int_2^4 (x^3+4)dx = \int_2^4 x^3\,dx + 4\int_2^4 dx = 60 + 4(2) = 68$

40. $\int_2^4 (6 + 2x - x^3)dx = 6\int_2^4 dx + 2\int_2^4 x\,dx - \int_2^4 x^3\,dx$

$= 6(2) + 2(6) - 60 = -36$

42. (a) $\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx = 4 + (-1) = 3$

(b) $\int_6^3 f(x)\,dx = -\int_3^6 f(x)\,dx = -(-1) = 1$

(c) $\int_3^3 f(x)\,dx = 0$

(d) $\int_3^6 -5f(x)\,dx = -5\int_3^6 f(x)\,dx = -5(-1) = 5$

44. (a) $\int_{-1}^0 f(x)\,dx = \int_{-1}^1 f(x)\,dx - \int_0^1 f(x)\,dx = 0 - 5 = -5$

(b) $\int_0^1 f(x)\,dx - \int_1^0 f(x)\,dx = 5 - (-5) = 10$

(c) $\int_{-1}^1 3f(x)\,dx = 3\int_{-1}^1 f(x)\,dx = 3(0) = 0$

(d) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3(5) = 15$

46. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$

(b) $\int_{-2}^3 f(x+2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^5 f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even)

(d) $\int_{-5}^5 f(x)\,dx = 0$ (f odd)

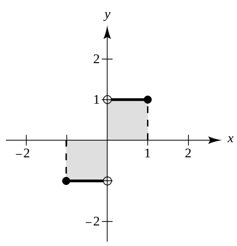
48. The right endpoint approximation will be less than the actual area: $<$

50. The average of Exercise 39 and Exercise 40 consists of a trapezoidal approximation, and is greater than the exact area: $>$

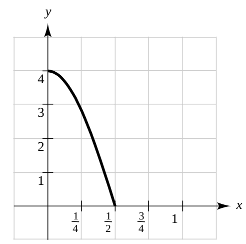
52. $f(x) = |x|/x$ is integrable on $[-1, 1]$, but is not continuous on $[-1, 1]$. There is discontinuity at $x = 0$. To see that

$$\int_{-1}^1 \frac{|x|}{x}\,dx$$

is integrable, sketch a graph of the region bounded by $f(x) = |x|/x$ and the x-axis for $-1 \quad x \quad 1$. You see that the integral equals 0.

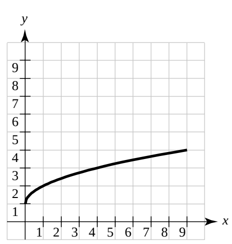


54.



b. $A \approx \frac{4}{3}$ square units

56.



c. Area ≈ 27.

58. $\int_0^3 \frac{5}{x^2 + 1}\,dx$

n	4	8	12	16	20
$L(n)$	7.9224	7.0855	6.8062	6.6662	6.5822
$M(n)$	6.2485	6.2470	7.2460	6.2457	6.2455
$R(n)$	4.5474	5.3980	5.6812	5.8225	5.9072

60. $\int_0^3 x \sin x\,dx$

n	4	8	12	16	20
$L(n)$	2.8186	2.9985	3.0434	3.0631	3.0740
$M(n)$	3.1784	3.1277	3.1185	3.1152	3.1138
$R(n)$	3.1361	3.1573	3.1493	3.1425	3.1375

62. False

$$\int_0^1 x\sqrt{x}\,dx \neq \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$$

64. True

66. False

$$\int_{-2}^4 x\,dx = 6$$

68. $f(x) = \sin x, [0, 2\pi]$

$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{3}, x_3 = \pi, x_4 = 2\pi$

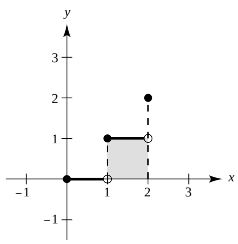
$\Delta x_1 = \frac{\pi}{4}, \Delta x_2 = \frac{\pi}{12}, \Delta x_3 = \frac{2\pi}{3}, \Delta x_4 = \pi$

$c_1 = \frac{\pi}{6}, c_2 = \frac{\pi}{3}, c_3 = \frac{2\pi}{3}, c_4 = \frac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

70. To find $\int_0^2 [\![x]\!]\,dx$, use a geometric approach.



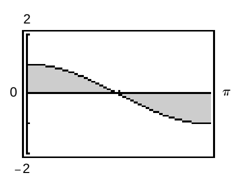
Thus,

$$\int_0^2 [\![x]\!]\,dx = 1(2-1) = 1.$$

Section 4.4 The Fundamental Theorem of Calculus

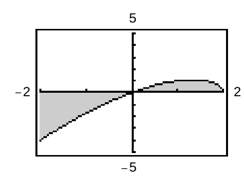
2. $f(x) = \cos x$

$$\int_0^{\pi} \cos x\,dx = 0$$



4. $f(x) = x\sqrt{2-x}$

$\int_{-2}^{2} x\sqrt{2-x}\,dx$ is negative.



6. $\int_2^7 3\,dv = \Big[3v\Big]_2^7 = 3(7) - 3(2) = 15$

8. $\int_2^5 (-3v + 4)\,dv = \left[-\frac{3}{2}v^2 + 4v\right]_2^5 = \left(-\frac{75}{2} + 20\right) - (-6 + 8) = -\frac{39}{2}$

10. $\int_1^3 (3x^2 + 5x - 4)dx = \left[x^3 + \frac{5x^2}{2} - 4x\right]_1^3 = \left(27 + \frac{45}{2} - 12\right) - \left(1 + \frac{5}{2} - 4\right)$

$$= 38$$

12. $\int_{-1}^{1} (t^3 - 9t)\,dt = \left[\frac{1}{4}t^4 - \frac{9}{2}t^2\right]_{-1}^{1} = \left(\frac{1}{4} - \frac{9}{2}\right) - \left(\frac{1}{4} - \frac{9}{2}\right) = 0$

14. $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du = \left[\frac{u^2}{2} + \frac{1}{u}\right]_{-2}^{-1} = \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2$

16. $\int_{-3}^{3} v^{1/3}\,dv = \left[\frac{3}{4}v^{4/3}\right]_{-3}^{3} = \frac{3}{4}\left[\left(\sqrt[3]{-3}\right)^4\right] - \left(\sqrt[3]{-3}\right)^4\right] = 0$

18. $\int_{1}^{8} \sqrt{\frac{2}{x}}\,dx = \sqrt{2}\int_{1}^{8} x^{-1/2}\,dx = \left[\sqrt{2}(2)x^{1/2}\right]_{1}^{8} = \left[2\sqrt{2x}\right]_{1}^{8} = 8 - 2\sqrt{2}$

20. $\int_{0}^{2} (2-t)\sqrt{t}\,dt = \int_{0}^{2} (2t^{1/2} - t^{3/2})\,dt = \left[\frac{4}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_{0}^{2} = \left[\frac{t\sqrt{t}}{15}(20 - 6t)\right]_{0}^{2} = \frac{2\sqrt{2}}{15}(20 - 12) = \frac{16\sqrt{2}}{15}$

22. $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\,dx = \frac{1}{2}\int_{-8}^{-1} (x^{2/3} - x^{5/3})\,dx$

$$= \frac{1}{2}\left[\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3}\right]_{-8}^{-1} = \left[\frac{x^{5/3}}{80}(24 - 15x)\right]_{-8}^{-1} = -\frac{1}{80}(39) + \frac{32}{80}(144) = \frac{4569}{80}$$

24. $\int_{1}^{4} (3 - 1x - 31)\,dx = \int_{1}^{3} [3 + (x - 3)]\,dx + \int_{3}^{4} [3 - (x - 3)]\,dx$

$$= \int_{1}^{3} x\,dx + \int_{3}^{4} (6 - x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{1}^{3} + \left[6x - \frac{x^2}{2}\right]_{3}^{4}$$

$$= \left(\frac{9}{2} - \frac{1}{2}\right) + \left[(24 - 8) - \left(18 - \frac{9}{2}\right)\right]$$

$$= 4 + 16 - 18 + \frac{9}{2} = \frac{13}{2}$$

26. $\int_{0}^{4} |x^2 - 4x + 3|\,dx = \int_{0}^{1} (x^2 - 4x + 3)\,dx - \int_{1}^{3} (x^2 - 4x + 3)\,dx + \int_{3}^{4} (x^2 - 4x + 3)\,dx$ (split up the integral at the zeros $x = 1, 3$)

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{0}^{1} - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{1}^{3} + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{3}^{4}$$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)$$

$$= \frac{4}{3} - 0 + \frac{4}{3} + \frac{4}{3} - 0 = 4$$

28. $\int_{0}^{\pi/4} \frac{1 - \sin^2\theta}{\cos^2\theta}\,d\theta = \int_{0}^{\pi/4} d\theta = \left[\theta\right]_{0}^{\pi/4} = \frac{\pi}{4}$

30. $\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)\,dx = \left[2x + \cot x\right]_{\pi/4}^{\pi/2} = (\pi + 0) - \left(\frac{\pi}{2} + 1\right) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$

32. $\int_{-\pi/2}^{\pi/2} (2t + \cos t)\,dt = \left[t^2 + \sin t\right]_{-\pi/2}^{\pi/2} = \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) = 2$

34. $P = \frac{2}{\pi}\int_{0}^{\pi/2} \sin\theta\,d\theta = \left[-\frac{2}{\pi}\cos\theta\right]_{0}^{\pi/2} = -\frac{2}{\pi}(0 - 1) = \frac{2}{\pi} \approx 63.7\%$

36. $A = \int_{-1}^{1} (1 - x^4)\,dx = \left[x - \frac{1}{5}x^5\right]_{-1}^{1} = \frac{8}{5}$

38. $A = \int_{1}^{2} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{1}^{2} = -\frac{1}{2} + 1 = \frac{1}{2}$

40. $A = \int_{0}^{\pi} (x + \sin x)\,dx = \left[\frac{x^2}{2} - \cos x\right]_{0}^{\pi} = \frac{\pi^2}{2} + 2 = \frac{\pi^2 + 4}{2}$

42. Since y 0 on [0. 8],

$$\text{Area} = \int_0^8 (1 + x^{1/3})\,dx = \left[x + \frac{3}{4}x^{4/3}\right]_0^8 = 8 + \frac{3}{4}(16) = 20$$

44. Since y 0 on [0, 3],

$$A = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}.$$

46. $\displaystyle\int_1^3 \frac{9}{x^3}\,dx = \left[-\frac{9}{2x^2}\right]_1^3 = -\frac{1}{2} + \frac{9}{2} = 4$

$$f(c)(3 - 1) = 4$$

$$\frac{9}{c^3} = 2$$

$$c^3 = \frac{9}{2}$$

$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510$$

48. $\displaystyle\int_{-\pi/3}^{\pi/3} \cos x\,dx = \Big[\sin x\Big]_{-\pi/3}^{\pi/3} = \sqrt{3}$

$$f(c)\left[\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right] = \sqrt{3}$$

$$\cos c = \frac{3\sqrt{3}}{2\pi}$$

$$c \approx \pm 0.5971$$

50. $\displaystyle\frac{1}{3 - 1}\int_1^3 \frac{4(x^2 + 1)}{x^2}\,dx = 2\int_1^3 (1 + x^{-2})dx = 2\left[x - \frac{1}{x}\right]_1^3$

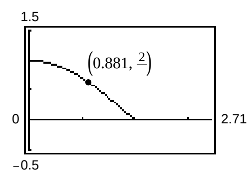
$$= 2\left(3 - \frac{1}{3}\right) = \frac{16}{3}$$

52. $\displaystyle\frac{1}{(\pi/2) - 0}\int_0^{\pi/2} \cos x\,dx = \left[\frac{2}{\pi}\sin x\right]_0^{\pi/2} = \frac{2}{\pi}$

$$\text{Average value} = \frac{2}{\pi}$$

$$\cos x = \frac{2}{\pi}$$

$$x \approx 0.881$$

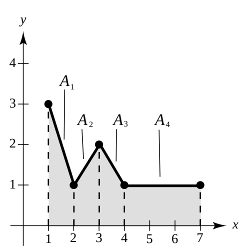


54. (a) $\displaystyle\int_1^7 f(x)\,dx = \text{Sum of the areas}$

$$= A_1 + A_2 + A_3 + A_4$$

$$= \frac{1}{2}(3 + 1) + \frac{1}{2}(1 + 2) + \frac{1}{2}(2 + 1) + (3)(1)$$

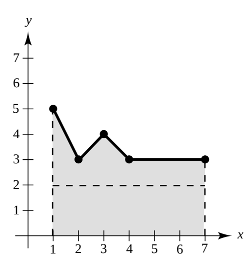
$$= 8$$



(b) $\displaystyle\text{Average value} = \frac{\int_1^7 f(x)\,dx}{7 - 1} = \frac{8}{6} = \frac{4}{3}$

(c) $A = 8 + (6)(2) = 20$

$$\text{Average value} = \frac{20}{6} = \frac{10}{3}$$



56. $\int_2^6 f(x)\,dx = (\text{area or region } B) = \int_0^6 f(x)\,dx - \int_0^2 f(x)\,dx$

$= 3.5 - (-1.5) = 5.0$

58. $\int_0^2 -2f(x)\,dx = -2\int_0^2 f(x)\,dx = -2(-1.5) = 3.0$

60. Average value $= \frac{1}{6}\int_0^6 f(x)\,dx = \frac{1}{6}(3.5) = 0.5833$

62. $\frac{1}{R-0}\int_0^R k(R^2 - r^2)\,dr = \frac{k}{R}\left[R^2 r - \frac{r^3}{3}\right]_0^R = \frac{2kR^2}{3}$

64. $P = 5\left(\sqrt{t} + 30\right)$

(a)

t	1	2	3	4	5	6
P	155	157.071	158.660	160	161.180	162.247

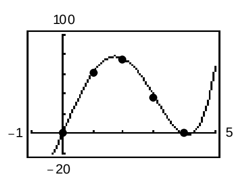
Average profit $\approx \frac{1}{6}(155 + 157.071 + 158.660 + 160 + 161.180 + 162.247) = \frac{954.158}{6} \approx 159.026$

(b) $\frac{1}{6}\int_{0.5}^{6.5} 5\left(\sqrt{t} + 30\right)dt = \frac{1}{6}\left[5\left(\frac{2}{3}t^{3/2} + 30t\right)\right]_{5.0}^{6.5} \approx \frac{954.061}{6} \approx 159.010$

(c) The definite integral yields a better approximation.

66. (a) $R = 2.33t^4 - 14.67t^3 + 3.67t^2 + 70.67t$

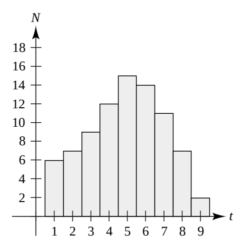
(b)



(c) $\int_0^4 R(t)\,dt = \left[\frac{2.33t^5}{5} - \frac{14.67t^4}{4} + \frac{3.67t^3}{3} + \frac{70.67t^2}{2}\right]_0^4$

$= 181.957$

68. (a) histogram

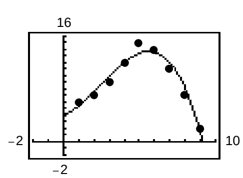


(b) $[6 + 7 + 9 + 12 + 15 + 14 + 11 + 7 + 2]60 = (83)60 = 4980$ customers

(c) Using a graphing utility, you obtain

$N(t) = -0.084175t^3 + 0.63492t^2 + 0.79052 + 4.10317.$

(d)



(e) $\int_0^9 N(t)\,dt \approx 85.162$

The estimated number of customers is $(85.162)(60) \approx 5110$.

(f) Between 3 P.M. and 7 P.M., the number of customers is approximately $\left(\int_3^7 N(t)\,dt\right)(60) \approx (50.28)(60) \approx 3017$.

Hence, $3017/240 \approx 12.6$ per minute.

70. $F(x) = \int_2^x (t^3 + 2t - 2)\,dt = \left[\frac{t^4}{4} + t^2 - 2t\right]_2^x$

$= \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4)$

$= \frac{x^4}{4} + x^2 - 2x - 4$

$F(2) = 4 + 4 - 4 - 4 = 0$ $\left[\text{Note: } F(2) = \int_2^2 (t^3 + 2t - 2)\,dt = 0\right]$

$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25$

$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068$

72. $F(x) = \int_2^x \frac{-2}{t^3}\,dt = -\int_2^x 2t^{-3}\,dt = \frac{1}{t^2}\Big]_2^x = \frac{1}{x^2} - \frac{1}{4}$

$F(2) = \frac{1}{4} - \frac{1}{4} = 0$

$F(5) = \frac{1}{25} - \frac{1}{4} = -\frac{21}{100} = -0.21$

$F(8) = \frac{1}{64} - \frac{1}{4} = -\frac{15}{64}$

74. $F(x) = \int_0^x \sin\theta\,d\theta = -\cos\theta\Big]_0^x = -\cos x + \cos 0 = 1 - \cos x$

$F(2) = 1 - \cos 2 \approx 1.4161$

$F(5) = 1 - \cos 5 \approx 0.7163$

$F(8) = 1 - \cos 8 \approx 1.1455$

76. (a) $\int_0^x t(t^2 + 1)\,dt = \int_0^x (t^3 + t)\,dt = \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_0^x = \frac{1}{4}x^4 + \frac{1}{2}x^2 = \frac{x^2}{4}(x^2 + 2)$

(b) $\frac{d}{dx}\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right] = x^3 + x = x(x^2 + 1)$

78. (a) $\int_4^x \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^x = \frac{2}{3}x^{3/2} - \frac{16}{3} = \frac{2}{3}(x^{3/2} - 8)$

(b) $\frac{d}{dx}\left[\frac{2}{3}x^{3/2} - \frac{16}{3}\right] = x^{1/2} = \sqrt{x}$

80. (a) $\int_{\pi/3}^x \sec t \tan t\,dt = \left[\sec t\right]_{\pi/3}^x = \sec x - 2$

(b) $\frac{d}{dx}[\sec x - 2] = \sec x \tan x$

82. $F(x) = \int_1^x \frac{t^2}{t^2 + 1}\,dt$

$F'(x) = \frac{x^2}{x^2 + 1}$

84. $F(x) = \int_1^x \sqrt[4]{t}\,dt$

$F'(x) = \sqrt[4]{x}$

86. $F(x) = \int_0^x \sec^3 t\,dt$

$F'(x) = \sec^3 x$

88. $F(x) = \int_{-x}^{x} t^3\,dt = \left[\frac{t^4}{4}\right]_{-x}^{x} = 0$

$F'(x) = 0$

Alternate solution

$$F(x) = \int_{-x}^{x} t^3\,dt$$
$$= \int_{-x}^{0} t^3\,dt + \int_{0}^{x} t^3\,dt$$
$$= -\int_{0}^{-x} t^3\,dt + \int_{0}^{x} t^3\,dt$$
$$F'(x) = -(-x)^3(-1) + (x^3) = 0$$

90. $F(x) = \int_{2}^{x^2} t^{-3}\,dt = \left[\frac{t^{-2}}{-2}\right]_2^{x^2} = \left[-\frac{1}{2t^2}\right]_2^{x^2} = \frac{-1}{2x^4} + \frac{1}{8}$ $\quad F'(x) = 2x^{-5}$

Alternate solution: $F'(x) = (x^2)^{-3}(2x) = 2x^{-5}$

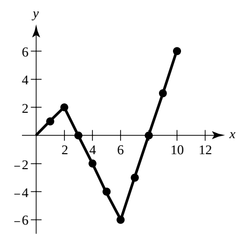
92. $F(x) = \int_{0}^{x^2} \sin\theta^2\,d\theta$

$F'(x) = \sin(x^2)^2\,(2x) = 2x\sin x^4$

94. (a)

x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(b)



(c) Minimum of g at $(6, -6)$.

(d) Minimum at $(10, 6)$. Relative maximum at $(2, 2)$.

(e) On $[6, 10]$ g increases at a rate of $\frac{12}{4} = 3$.

(f) Zeros of g: $x = 3, x = 8$.

96. (a) $g(t) = 4 - \frac{4}{t^2}$

$\lim_{t \to \infty} g(t) = 4$

Horizontal asymptote: $y = 4$

(b) $A(x) = \int_{1}^{x} \left(4 - \frac{4}{t^2}\right) dt$

$$= \left[4t + \frac{4}{t}\right]_1^x = 4x + \frac{4}{x} - 8$$
$$= \frac{4x^2 - 8x + 4}{x} = \frac{4(x-1)^2}{x}$$

$$\lim_{x \to \infty} A(x) = \lim_{x \to \infty}\left(4x + \frac{4}{x} - 8\right) = \infty + 0 - 8 = \infty$$

The graph of $A(x)$ does not have a horizontal asymptote.

98. True

100. Let $F(t)$ be an antiderivative of $f(t)$. Then,

$$\int_{u(x)}^{v(x)} f(t)\,dt = \left[F(t)\right]_{u(x)}^{v(x)} = F(v(x)) - F(u(x))$$
$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = \frac{d}{dx}[F(v(x)) - F(u(x))]$$
$$= F'(v(x))v'(x) - F'(u(x))u'(x)$$
$$= f(v(x))v'(x) - f(u(x))u'(x).$$

102. $G(x) = \int_0^x \left[s \int_0^s f(t)\, dt \right] ds$

(a) $G(0) = \int_0^0 \left[s \int_0^s f(t)\, dt \right] ds = 0$

(b) Let $F(s) = s \int_0^s f(t)\, dt$.

$G(x) = \int_0^x F(s)\, ds$

$G'(x) = F(x) = x \int_0^x f(t)\, dt$

$G'(0) = 0 \int_0^0 f(t)\, dt = 0$

(c) $G''(x) = x \cdot f(x) + \int_0^x f(t)\, dt$

(d) $G''(0) = 0 \cdot f(0) + \int_0^0 f(t)\, dt = 0$

104. $x(t) = (t-1)(t-3)^2 = t^3 - 7t^2 + 15t - 9$

$x'(t) = 3t^2 - 14t + 15$

Using a graphing utility,

Total distance $= \int_0^5 |x'(t)|\, dt \approx 27.37$ units

Section 4.5 Integration by Substitution

$\int f(g(x))g'(x)\, dx$	$u = g(x)$	$du = g'(x)\, dx$
2. $\int x^2 \sqrt{x^3+1}\, dx$	$x^3 + 1$	$3x^2\, dx$
4. $\int \sec 2x \tan 2x\, dx$	$2x$	$2\, dx$
6. $\int \frac{\cos x}{\sin^2 x}\, dx$	$\sin x$	$\cos x\, dx$

8. $\int (x^2 - 9)^3(2x)\, dx = \frac{(x^2-9)^4}{4} + C$

Check: $\frac{d}{dx}\left[\frac{(x^2-9)^4}{4} + C \right] = \frac{4(x^2-9)^3}{4}(2x) = (x^2-9)^3(2x)$

10. $\int (1 - 2x^2)^{1/3}(-4x)\, dx = \frac{3}{4}(1 - 2x^2)^{4/3} + C$

Check: $\frac{d}{dx}\left[\frac{3}{4}(1 - 2x^2)^{4/3} + C \right] = \frac{3}{4} \cdot \frac{4}{3}(1 - 2x^2)^{1/3}(-4x) = (1 - 2x^2)^{1/3}(-4x)$

12. $\int x^2(x^3+5)^4\, dx = \frac{1}{3}\int (x^3+5)^4(3x^2)\, dx = \frac{1}{3}\frac{(x^3+5)^5}{5} + C = \frac{(x^3+5)^5}{15} + C$

Check: $\frac{d}{dx}\left[\frac{(x^3+5)^5}{15} + C \right] = \frac{5(x^3+5)^4(3x^2)}{15} = (x^3+5)^4x^2$

14. $\int x(4x^2+3)^3\, dx = \frac{1}{8}\int (4x^2+3)^3(8x)\, dx = \frac{1}{8}\left[\frac{(4x^2+3)^4}{4} \right] + C = \frac{(4x^2+3)^4}{32} + C$

Check: $\frac{d}{dx}\left[\frac{(4x^2+3)^4}{32} + C \right] = \frac{4(4x^2+3)^3(8x)}{32} = x(4x^2+3)^3$

16. $$\int t^3\sqrt{t^4+5}\,dt = \frac{1}{4}\int (t^4+5)^{1/2}(4t^3)\,dt = \frac{1}{4}\frac{(t^4+5)^{3/2}}{3/2} + C = \frac{1}{6}(t^4+5)^{3/2} + C$$

Check: $$\frac{d}{dt}\left[\frac{1}{6}(t^4+5)^{3/2} + C\right] = \frac{1}{6}\cdot\frac{3}{2}(t^4+5)^{1/2}(4t^3) = (t^4+5)^{1/2}(t^3)$$

18. $$\int u^2\sqrt{u^3+2}\,du = \frac{1}{3}\int (u^3+2)^{1/2}(3u^2)\,du = \frac{1}{3}\frac{(u^3+2)^{3/2}}{3/2} + C = \frac{2(u^3+2)^{3/2}}{9} + C$$

Check: $$\frac{d}{du}\left[\frac{2(u^3+2)^{3/2}}{9} + C\right] = \frac{2}{9}\cdot\frac{3}{2}(u^3+2)^{1/2}(3u^2) = (u^3+2)^{1/2}(u^2)$$

20. $$\int \frac{x^3}{(1+x^4)^2}\,dx = \frac{1}{4}\int (1+x^4)^{-2}(4x^3)\,dx = -\frac{1}{4}(1+x^4)^{-1} + C = \frac{-1}{4(1+x^4)} + C$$

Check: $$\frac{d}{dx}\left[\frac{-1}{4(1+x^4)} + C\right] = \frac{1}{4}(1+x^4)^{-2}(4x^3) = \frac{x^3}{(1+x^4)^2}$$

22. $$\int \frac{x^2}{(16-x^3)^2}\,dx = -\frac{1}{3}\int (16-x^3)^{-2}(-3x^2)\,dx = -\frac{1}{3}\left[\frac{(16-x^3)^{-1}}{-1}\right] + C = \frac{1}{3(16-x^3)} + C$$

Check: $$\frac{d}{dx}\left[\frac{1}{3(16-x^3)} + C\right] = \frac{1}{3}(-1)(16-x^3)^{-2}(3x^2) = \frac{x^2}{(16-x^3)^2}$$

24. $$\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int (1+x^4)^{-1/2}(4x^3)\,dx = \frac{1}{4}\frac{(1+x^4)^{1/2}}{1/2} + C = \frac{\sqrt{1+x^4}}{2} + C$$

Check: $$\frac{d}{dx}\left[\frac{\sqrt{1+x^4}}{2} + C\right] = \frac{1}{2}\cdot\frac{1}{2}(1+x^4)^{-1/2}(4x^3) = \frac{x^3}{\sqrt{1+x^4}}$$

26. $$\int\left[x^2 + \frac{1}{(3x)^2}\right]dx = \int\left(x^2 + \frac{1}{9}x^{-2}\right)dx = \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{3} - \frac{1}{9x} + C = \frac{3x^4-1}{9x} + C$$

Check: $$\frac{d}{dx}\left[\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right] = x^2 + \frac{1}{9}x^{-2} = x^2 + \frac{1}{(3x)^2}$$

28. $$\int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx = \frac{1}{2}\left(\frac{x^{1/2}}{1/2}\right) + C = \sqrt{x} + C$$

Check: $$\frac{d}{dx}\left[\sqrt{x} + C\right] = \frac{1}{2\sqrt{x}}$$

30. $$\int \frac{t+2t^2}{\sqrt{t}}\,dt = \int (t^{1/2} + 2t^{3/2})\,dt = \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C = \frac{2}{15}t^{3/2}(5+6t) + C$$

Check: $$\frac{d}{dt}\left[\frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/2} + C\right] = t^{1/2} + 2t^{3/2} = \frac{t+2t^2}{\sqrt{t}}$$

32. $$\int\left(\frac{t^3}{3} + \frac{1}{4t^2}\right)dt = \int\left(\frac{1}{3}t^3 + \frac{1}{4}t^{-2}\right)dt = \frac{1}{3}\left(\frac{t^4}{4}\right) + \frac{1}{4}\left(\frac{t^{-1}}{-1}\right) + C = \frac{1}{12}t^4 - \frac{1}{4t} + C$$

Check: $$\frac{d}{dt}\left[\frac{1}{12}t^4 - \frac{1}{4t} + C\right] = \frac{1}{3}t^3 + \frac{1}{4t^2}$$

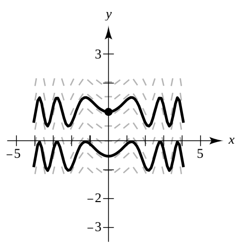
34. $\int 2\pi y(8 - y^{3/2})\,dy = 2\pi\int(8y - y^{5/2})\,dy = 2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C = \frac{4\pi y^2}{7}(14 - y^{3/2}) + C$

Check: $\frac{d}{dy}\left[\frac{4\pi y^2}{7}(14 - y^{3/2}) + C\right] = \frac{d}{dy}\left[2\pi\left(4y^2 - \frac{2}{7}y^{7/2}\right) + C\right] = 16\pi y - 2\pi y^{5/2} = (2\pi y)(8 - y^{3/2})$

36. $$\begin{aligned} y &= \int \frac{10x^2}{\sqrt{1 + x^3}}\,dx \\ &= \frac{10}{3}\int(1 + x^3)^{-1/2}(3x^2)\,dx \\ &= \frac{10}{3}\left[\frac{(1 + x^3)^{1/2}}{1/2}\right] + C \\ &= \frac{20}{3}\sqrt{1 + x^3} + C \end{aligned}$$

38. $$\begin{aligned} y &= \int \frac{x - 4}{\sqrt{x^2 - 8x + 1}}\,dx \\ &= \frac{1}{2}\int(x^2 - 8x + 1)^{-1/2}(2x - 8)\,dx \\ &= \frac{1}{2}\left[\frac{(x^2 - 8x + 1)^{1/2}}{1/2}\right] + C \\ &= \sqrt{x^2 - 8x + 1} + C \end{aligned}$$

40. (a)



(b) $\frac{dy}{dx} = x\cos x^2,\ (0, 1)$

$$y = \int x\cos x^2\,dx = \frac{1}{2}\int \cos(x^2)2x\,dx$$

$$= \frac{1}{2}\sin(x^2) + C$$

$(0, 1)$: $1 = \frac{1}{2}\sin(0) + C \qquad C = 1$

$$y = \frac{1}{2}\sin(x^2) + 1$$

42. $\int 4x^3\sin x^4\,dx = \int \sin x^4(4x^3)\,dx = -\cos x^4 + C$

44. $\int \cos 6x\,dx = \frac{1}{6}\int(\cos 6x)(6)\,dx = \frac{1}{6}\sin 6x + C$

46. $\int x\sin x^2\,dx = \frac{1}{2}\int(\sin x^2)(2x)\,dx = -\frac{1}{2}\cos x^2 + C$

48. $\int \sec(1 - x)\tan(1 - x)\,dx = -\int[\sec(1 - x)\tan(1 - x)](-1)\,dx = -\sec(1 - x) + C$

50. $\int \sqrt{\tan x}\sec^2 x\,dx = \frac{(\tan x)^{3/2}}{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$

52. $\int \frac{\sin x}{\cos^3 x}\,dx = -\int(\cos x)^{-3}(-\sin x)\,dx = -\frac{(\cos x)^{-2}}{-2} + C = \frac{1}{2\cos^2 x} + C = \frac{1}{2}\sec^2 x + C$

54. $\int \csc^2\left(\frac{x}{2}\right)dx = 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)dx = -2\cot\left(\frac{x}{2}\right) + C$

56. $f(x) = \int \pi\sec\pi x\tan\pi x\,dx = \sec\pi x + C$

Since $f(1/3) = 1 = \sec(\pi/3) + C$, $C = -1$. Thus

$f(x) = \sec\pi x - 1.$

58. $u = 2x + 1, x = \frac{1}{2}(u - 1), dx = \frac{1}{2}\,du$

$$\begin{aligned}\int x\sqrt{2x+1}\,dx &= \int \frac{1}{2}(u-1)\sqrt{u}\,\frac{1}{2}\,du\\ &= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du\\ &= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C\\ &= \frac{u^{3/2}}{30}(3u - 5) + C\\ &= \frac{1}{30}(2x+1)^{3/2}[3(2x+1) - 5] + C\\ &= \frac{1}{30}(2x+1)^{3/2}(6x - 2) + C\\ &= \frac{1}{15}(2x+1)^{3/2}(3x - 1) + C\end{aligned}$$

60. $u = 2 - x, x = 2 - u, dx = -du$

$$\begin{aligned}\int (x+1)\sqrt{2-x}\,dx &= -\int (3-u)\sqrt{u}\,du\\ &= -\int (3u^{1/2} - u^{3/2})\,du\\ &= -\left(2u^{3/2} - \frac{2}{5}u^{5/2}\right) + C\\ &= -\frac{2u^{3/2}}{5}(5 - u) + C\\ &= -\frac{2}{5}(2-x)^{3/2}[5 - (2-x)] + C\\ &= -\frac{2}{5}(2-x)^{3/2}(x+3) + C\end{aligned}$$

62. Let $u = x + 4, x = u - 4, du = dx$.

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x+4}}\,dx &= \int \frac{2(u-4)+1}{\sqrt{u}}\,du\\ &= \int (2u^{1/2} - 7u^{-1/2})\,du\\ &= \frac{4}{3}u^{3/2} - 14u^{1/2} + C\\ &= \frac{2}{3}u^{1/2}(2u - 21) + C\\ &= \frac{2}{3}\sqrt{x+4}[2(x+4) - 21] + C\\ &= \frac{2}{3}\sqrt{x+4}(2x - 13) + C\end{aligned}$$

64. $u = t - 4, t = u + 4, dt = du$

$$\begin{aligned}\int t\sqrt[3]{t-4}\,dt &= \int (u+4)u^{1/3}\,du\\ &= \int (u^{4/3} + 4u^{1/3})\,du\\ &= \frac{3}{7}u^{7/3} + 3u^{4/3} + C\\ &= \frac{3u^{4/3}}{7}(u + 7) + C\\ &= \frac{3}{7}(t-4)^{4/3}[(t-4) + 7] + C\\ &= \frac{3}{7}(t-4)^{4/3}(t+3) + C\end{aligned}$$

66. Let $u = x^3 + 8, du = 3x^2\,dx$.

$$\begin{aligned}\int_{-2}^{4} x^2(x^3+8)^2\,dx &= \frac{1}{3}\int_{-2}^{4} (x^3+8)^2(3x^2)\,dx = \left[\frac{1}{3}\,\frac{(x^3+8)^3}{3}\right]_{-2}^{4}\\ &= \frac{1}{9}[(64+8)^3 - (-8+8)^3] = 41{,}472\end{aligned}$$

68. Let $u = 1 - x^2, du = -2x\,dx$.

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_0^1 (1-x^2)^{1/2}(-2x)\,dx = \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_0^1 = 0 + \frac{1}{3} = \frac{1}{3}$$

70. Let $u = 1 + 2x^2, du = 4x\,dx$.

$$\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx = \frac{1}{4}\int_0^2 (1+2x^2)^{-1/2}(4x)\,dx = \left[\frac{1}{2}\sqrt{1+2x^2}\right]_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

72. Let $u = 4 + x^2$, $du = 2x\,dx$.

$$\int_0^2 x\sqrt[3]{4+x^2}\,dx = \frac{1}{2}\int_0^2 (4+x^2)^{1/3}(2x)\,dx = \left[\frac{3}{8}(4+x^2)^{4/3}\right]_0^2 = \frac{3}{8}(8^{4/3} - 4^{4/3}) = 6 - \frac{3}{2}\sqrt[3]{4} \approx 3.619$$

74. Let $u = 2x - 1$, $du = 2\,dx$, $x = \frac{1}{2}(u+1)$.

When $x = 1$, $u = 1$. When $x = 5$, $u = 9$.

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^9 \frac{1/2(u+1)}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{4}\int_1^9 (u^{1/2} + u^{-1/2})\,du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^9$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}(27) + 2(3)\right) - \left(\frac{2}{3} + 2\right)\right]$$

$$= \frac{16}{3}$$

76. $$\int_{\pi/3}^{\pi/2} (x + \cos x)\,dx = \left[\frac{x^2}{2} + \sin x\right]_{\pi/3}^{\pi/2} = \left(\frac{\pi^2}{8} + 1\right) - \left(\frac{\pi^2}{18} + \frac{\sqrt{3}}{2}\right) = \frac{5\pi^2}{72} + \frac{2-\sqrt{3}}{2}$$

78. $u = x + 2$, $x = u - 2$, $dx = du$

When $x = -2$, $u = 0$. When $x = 6$, $u = 8$.

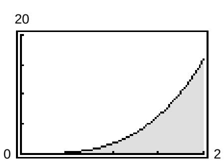
$$\text{Area} = \int_{-2}^6 x^2\sqrt[3]{x+2}\,dx = \int_0^8 (u-2)^2\sqrt[3]{u}\,du = \int_0^8 (u^{7/3} - 4u^{4/3} + 4u^{1/3})\,du = \left[\frac{3}{10}u^{10/3} - \frac{12}{7}u^{7/3} + 3u^{4/3}\right]_0^8 = \frac{4752}{35}$$

80. $$A = \int_0^\pi (\sin x + \cos 2x)\,dx = \left[-\cos x + \frac{1}{2}\sin 2x\right]_0^\pi = 2$$

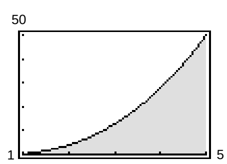
82. Let $u = 2x$, $du = 2\,dx$.

$$\text{Area} = \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\,dx = \frac{1}{2}\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x(2)\,dx = \left[-\frac{1}{2}\csc 2x\right]_{\pi/12}^{\pi/4} = \frac{1}{2}$$

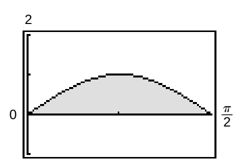
84. $\int_0^2 x^3\sqrt{x+2}\,dx \approx 7.581$



86. $\int_1^5 x^2\sqrt{x-1}\,dx \approx 67.505$



88. $\int_0^{\pi/2} \sin 2x\,dx \approx 1.0$



90. $$\int \sin x \cos x\,dx = \int (\sin x)^1(\cos x\,dx) = \frac{\sin^2 x}{2} + C_1$$

$$\int \sin x \cos x\,dx = -\int (\cos x)^1(-\sin x\,dx) = -\frac{\cos^2 x}{2} + C_2$$

$$-\frac{\cos^2 x}{2} + C_2 = -\frac{(1 - \sin^2 x)}{2} + C_2 = \frac{\sin^2 x}{2} - \frac{1}{2} + C_2$$

They differ by a constant: $C_2 = C_1 + \frac{1}{2}$.

92. $f(x) = \sin^2 x \cos x$ is even.

$$\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx = \int_0^{\pi/2} \sin^2 x(\cos x)\, dx = 2\left[\frac{\sin^3 x}{3}\right]_0^{\pi/2} = \frac{2}{3}$$

94. $f(x) = \sin x \cos x$ is odd.

$$\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx = 0$$

96. (a) $\int_{-\pi/4}^{\pi/4} \sin x\, dx = 0$ since $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\int_0^{\pi/4} \cos x\, dx = \left[2 \sin x\right]_0^{\pi/4} = \sqrt{2}$ since $\cos x$ is symmetric to the y-axis.

(c) $\int_{-\pi/2}^{\pi/2} \cos x\, dx = 2\int_0^{\pi/2} \cos x\, dx = \left[2 \sin x\right]_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x\, dx = 0$ since $\sin(-x)\cos(-x) = -\sin x \cos x$ and hence, is symmetric to the origin.

98. $\int_{-\pi}^{\pi} (\sin 3x + \cos 3x)\, dx = \int_{-\pi}^{\pi} \sin 3x\, dx + \int_{-\pi}^{\pi} \cos 3x\, dx = 0 + 2\int_0^{\pi} \cos 3x\, dx = \left[\frac{2}{3}\sin 3x\right]_0^{\pi} = 0$

100. If $u = 5 - x^2$, then $du = -2x\, dx$ and $\int x(5 - x^2)^3\, dx = -\frac{1}{2}\int (5 - x^2)^3(-2x)\, dx = -\frac{1}{2}\int u^3\, du$.

102. $\frac{dQ}{dt} = k(100 - t)^2$

$Q(t) = \int k(100 - t)^2\, dt = -\frac{k}{3}(100 - t)^3 + C$

$Q(100) = C = 0$

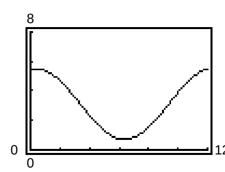
$Q(t) = -\frac{k}{3}(100 - t)^3$

$Q(0) = -\frac{k}{3}(100)^3 = 2{,}000{,}000 \qquad k = -6$

Thus, $Q(t) = 2(100 - t)^3$. When $t = 50$, $Q(50) =$ \$250,000.

104. $R = 3.121 + 2.399 \sin(0.524t + 1.377)$

(a)

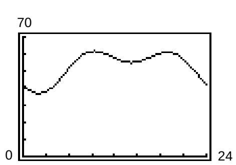


Relative minimum: (6.4, 0.7) or June
Relative maximum: (0.4, 5.5) or January

(b) $\int_0^{12} R(t)\, dt \approx 37.47$ inches

(c) $\frac{1}{3}\int_9^{12} R(t)\, dt \approx \frac{1}{3}(13) = 4.33$ inches

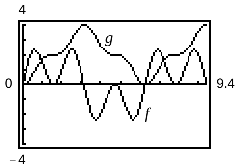
106. (a)



Maximum flow: $R \approx 61.713$ at $t = 9.36$.
[(18.861, 61.178) is a relative maximum.]

(b) Volume $= \int_0^{24} R(t)\, dt \approx 1272$ (5 thousand of gallons)

108. (a)

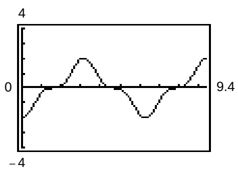


(b) g is nonnegative because the graph of f is positive at the beginning, and generally has more positive sections than negative ones.

(c) The points on g that correspond to the extrema of f are points of inflection of g.

(d) No, some zeros of f, like $x = \pi/2$, do not correspond to an extrema of g. The graph of g continues to increase after $x = \pi/2$ because f remains above the x-axis.

(e)



The graph of h is that of g shifted 2 units downward.

$$\begin{aligned} g(t) &= \int_0^t f(x)\,dx \\ &= \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^t f(x)\,dx = 2 + h(t). \end{aligned}$$

110. False

$$\int x(x^2+1)^2\,dx = \frac{1}{2}\int (x^2+1)(2x)\,dx = \frac{1}{4}(x^2+1)^2 + C$$

112. True

$$\int_a^b \sin x\,dx = \Big[-\cos x\Big]_a^b = -\cos b + \cos a = -\cos(b+2\pi) + \cos a = \int_a^{b+2\pi} \sin x\,dx$$

114. False

$$\int \sin^2 2x \cos 2x\,dx = \frac{1}{2}\int (\sin 2x)^2 (2\cos 2x)\,dx = \frac{1}{2}\frac{(\sin 2x)^3}{3} + C = \frac{1}{6}\sin^3 2x + C$$

116. Because f is odd, $f(-x) = -f(x)$. Then

$$\begin{aligned} \int_{-a}^a f(x)\,dx &= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \\ &= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx. \end{aligned}$$

Let $x = -u$, $dx = -du$ in the first integral.

When $x = 0$, $u = 0$. When $x = -a$, $u = a$.

$$\begin{aligned} \int_{-a}^1 f(x)\,dx &= -\int_0^a f(-u)(-du) + \int_0^a f(x)\,dx \\ &= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = 0 \end{aligned}$$

Section 4.6 Numerical Integration

2. Exact: $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx = \left[\frac{x^3}{6} + x\right]_0^1 = \frac{7}{6} \approx 1.1667$

Trapezoidal: $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx \approx \frac{1}{8}\left[1 + 2\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 2\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right) = \frac{75}{64} \approx 1.1719\right.$

Simpson's: $\int_0^1 \left(\frac{x^2}{2} + 1\right) dx \approx \frac{1}{12}\left[1 + 4\left(\frac{(1/4)^2}{2} + 1\right) + 2\left(\frac{(1/2)^2}{2} + 1\right) + 4\left(\frac{(3/4)^2}{2} + 1\right) + \left(\frac{1^2}{2} + 1\right)\right] = \frac{7}{6} \approx 1.1667$

4. Exact: $\int_1^2 \frac{1}{x^2}\, dx = \left[\frac{-1}{x}\right]_1^2 = 0.5000$

Trapezoidal: $\int_1^2 \frac{1}{x^2}\, dx \approx \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 2\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5090$

Simpson's: $\int_1^2 \frac{1}{x^2}\, dx \approx \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right)^2 + 2\left(\frac{4}{6}\right)^2 + 4\left(\frac{4}{7}\right)^2 + \frac{1}{4}\right] \approx 0.5004$

6. Exact: $\int_0^8 \sqrt[3]{x}\, dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

Trapezoidal: $\int_0^8 \sqrt[3]{x}\, dx \approx \frac{1}{2}[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2] \approx 11.7296$

Simpson's: $\int_0^8 \sqrt[3]{x}\, dx \approx \frac{1}{3}[0 + 4 + 2\sqrt[3]{2} + 4\sqrt[3]{3} + 2\sqrt[3]{4} + 4\sqrt[3]{5} + 2\sqrt[3]{6} + 4\sqrt[3]{7} + 2] \approx 11.8632$

8. Exact: $\int_1^3 (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_1^3 = 3 - \frac{11}{3} = -\frac{2}{3} \approx -0.6667$

Trapezoidal: $\int_1^3 (4 - x^2)\, dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] - 5\right\} = -0.7500$

Simpson's: $\int_1^3 (4 - x^2)\, dx \approx \frac{1}{6}\left[3 + 4\left(4 - \frac{9}{4}\right) + 0 + 4\left(4 - \frac{25}{4}\right) - 5\right] \approx -0.6667$

10. Exact: $\int_0^2 x\sqrt{x^2 + 1}\, dx = \frac{1}{3}\left[(x^2 + 1)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2} - 1) \approx 3.393$

Trapezoidal: $\int_0^2 x\sqrt{x^2 + 1}\, dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)\sqrt{(1/2)^2 + 1} + 2(1)\sqrt{1^2 + 1} + 2\left(\frac{3}{2}\right)\sqrt{(3/2)^2 + 1} + 2\sqrt{2^2 + 1}\right] \approx 3.457$

Simpson's: $\int_0^2 x\sqrt{x^2 + 1}\, dx \approx \frac{1}{6}\left[0 + 4\left(\frac{1}{2}\right)\sqrt{(1/2)^2 + 1} + 2(1)\sqrt{1^2 + 1} + 4\left(\frac{3}{2}\right)\sqrt{(3/2)^2 + 1} + 2\sqrt{2^2 + 1}\right] \approx 3.392$

12. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\, dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1 + (1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1 + 1^3}}\right) + 2\left(\frac{1}{\sqrt{1 + (3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Simpson's: $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\, dx \approx \frac{1}{6}\left[1 + 4\left(\frac{1}{\sqrt{1 + (1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1 + 1^3}}\right) + 4\left(\frac{1}{\sqrt{1 + (3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.405$

Graphing utility: 1.402

14. Trapezoidal: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{16}\left[\sqrt{\frac{\pi}{2}}(1) + 2\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 2\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.430$

Simpson's: $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx \approx \frac{\pi}{24}\left[\sqrt{\frac{\pi}{2}} + 4\sqrt{\frac{5\pi}{8}}\sin\left(\frac{5\pi}{8}\right) + 2\sqrt{\frac{3\pi}{4}}\sin\left(\frac{3\pi}{4}\right) + 4\sqrt{\frac{7\pi}{8}}\sin\left(\frac{7\pi}{8}\right) + 0\right] \approx 1.458$

Graphing utility: 1.458

16. Trapezoidal: $\int_{0}^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.271

Simpson's: $\int_{0}^{\sqrt{\pi/4}} \tan(x^2)\,dx \approx \frac{\sqrt{\pi/4}}{12}\left[\tan 0 + 4\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 4\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right]$

≈ 0.257

Graphing utility: 0.256

18. Trapezoidal: $\int_{0}^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{16}\left[\sqrt{2} + 2\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 2\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Simpson's: $\int_{0}^{\pi/2} \sqrt{1+\cos^2 x}\,dx \approx \frac{\pi}{24}\left[\sqrt{2} + 4\sqrt{1+\cos^2(\pi/8)} + 2\sqrt{1+\cos^2(\pi/4)} + 4\sqrt{1+\cos^2(3\pi/8)} + 1\right] \approx 1.910$

Graphing utility: 1.910

20. Trapezoidal: $\int_{0}^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Simpson's: $\int_{0}^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{12}\left[1 + \frac{4\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{4\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.852$

Graphing utility: 1.852

22. Trapezoidal: Linear polynomials

Simpson's: Quadratic polynomials

24. $f(x) = \frac{1}{x+1}$

$f'(x) = \frac{-1}{(x+1)^2}$

$f''(x) = \frac{2}{(x+1)^3}$

$f'''(x) = \frac{-6}{(x+1)^4}$

$f^{(4)}(x) = \frac{24}{(x+1)^5}$

(a) Trapezoidal: Error $\frac{(1-0)^2}{12(4^2)}(2) = \frac{1}{96} \approx 0.01$ since

$f''(x)$ is maximum in $[0, 1]$ when $x = 0$.

(b) Simpson's: Error $\frac{(1-0)^5}{180(4^4)}(24) = \frac{1}{1920} \approx 0.0005$

since $f^{(4)}(x)$ is maximum in $[0, 1]$ when $x = 0$.

26. $f''(x) = \frac{2}{(1+x)^3}$ in $[0, 1]$.

(a) $|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = 2$.

Trapezoidal: Error $\frac{1}{12n^2}(2) < 0.00001$, $n^2 > 16{,}666.67$, $n > 129.10$; let $n = 130$.

$f^{(4)}(x) = \frac{24}{(1+x)^5}$ in $[0, 1]$

(b) $|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = 24$.

Simpson's: Error $\frac{1}{180n^4}(24) < 0.00001$, $n^4 > 13{,}333.33$, $n > 10.75$; let $n = 12$. (In Simpson's Rule n must be even.)

28. $f(x) = (x+1)^{2/3}$

(a) $f''(x) = -\frac{2}{9(x+1)^{4/3}}$ in $[0, 2]$.

$|f''(x)|$ is maximum when $x = 0$ and $|f''(0)| = \frac{2}{9}$.

Trapezoidal: Error $\frac{8}{12n^4}\left(\frac{2}{9}\right) < 0.00001$, $n^2 > 14{,}814.81$, $n > 121.72$; let $n = 122$.

(b) $f^{(4)}(x) = -\frac{56}{81(x+1)^{10/3}}$ in $[0, 2]$

$|f^{(4)}(x)|$ is maximum when $x = 0$ and $|f^{(4)}(0)| = \frac{56}{81}$.

Simpson's: Error $\frac{32}{180n^4}\left(\frac{56}{81}\right) < 0.00001$, $n^4 > 12{,}290.81$, $n > 10.53$; let $n = 12$. (In Simpson's Rule n must be even.)

30. $f(x) = \sin(x^2)$

(a) $f''(x) = 2[-2x^2 \sin(x^2) + \cos(x^2)]$ in $[0, 1]$.

$|f''(x)|$ is maximum when $x = 1$ and $|f''(1)| \approx 2.2853$.

Trapezoidal: Error $\frac{(1-0)^3}{12n^2}(2.2853) < 0.00001$, $n^2 > 19{,}044.17$, $n > 138.00$; let $n = 139$.

(b) $f^{(4)}(x) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$ in $[0, 1]$

$|f^{(4)}(x)|$ is maximum when $x \approx 0.852$ and $|f^{(4)}(0.852)| \approx 28.4285$.

Simpson's: Error $\frac{(1-0)^5}{180n^4}(28.4285) < 0.00001$, $n^4 > 15{,}793.61$, $n > 11.21$; let $n = 12$.

32. The program will vary depending upon the computer or programmable calculator that you use.

34. $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.8739	0.7960	0.6239	0.7489	0.7709
8	0.8350	0.7892	0.7100	0.7725	0.7803
10	0.8261	0.7881	0.7261	0.7761	0.7818
12	0.8200	0.7875	0.7367	0.7783	0.7826
16	0.8121	0.7867	0.7496	0.7808	0.7836
20	0.8071	0.7864	0.7571	0.7821	0.7841

36. $f(x) = \dfrac{\sin x}{x}$ on $[1, 2]$.

n	$L(n)$	$M(n)$	$R(n)$	$T(n)$	$S(n)$
4	0.7070	0.6597	0.6103	0.6586	0.6593
8	0.6833	0.6594	0.6350	0.6592	0.6593
10	0.6786	0.6594	0.6399	0.6592	0.6593
12	0.6754	0.6594	0.6431	0.6593	0.6593
16	0.6714	0.6594	0.6472	0.6593	0.6593
20	0.6690	0.6593	0.6496	0.6593	0.6593

38. Simpson's Rule: $n = 8$

$$8\sqrt{3}\int_0^{\pi/2}\sqrt{1 - \frac{2}{3}\sin^2\theta\, d\theta} \approx \frac{\sqrt{3}\pi}{6}\left[\sqrt{1 - \frac{2}{3}\sin^2 0} + 4\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{16}} + 2\sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{8}} + \cdots + \sqrt{1 - \frac{2}{3}\sin^2\frac{\pi}{2}}\right]$$
$$\approx 17.476$$

40. (a) Trapezoidal:

$$\int_0^2 f(x)\,dx \approx \frac{2}{2(8)}[4.32 + 2(4.36) + 2(4.58) + 2(5.79) + 2(6.14) + 2(7.25) + 2(7.64) + 2(8.08) + 8.14] \approx 12.518$$

Simpson's:

$$\int_0^2 f(x)\,dx \approx \frac{2}{3(8)}[4.32 + 4(4.36) + 2(4.58) + 4(5.79) + 2(6.14) + 4(7.25) + 2(7.64) + 4(8.08) + 8.14] \approx 12.592$$

(b) Using a graphing utility,

$$y = -1.3727x^3 + 4.0092x^2 - 0.6202x + 4.2844$$

Integrating, $\displaystyle\int_0^2 y\,dx \approx 12.53$

42. Simpson's Rule: $n = 6$

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\,dx \approx \frac{4}{3(6)}\left[1 + \frac{4}{1 + (1/6)^2} + \frac{2}{1 + (2/6)^2} + \frac{4}{1 + (3/6)^2} + \frac{2}{1 + (4/6)^2} + \frac{4}{1 + (5/6)^2} + \frac{1}{2}\right]$$
$$\approx 3.14159$$

44. Area $\approx \dfrac{120}{2(12)}[75 + 2(81) + 2(84) + 2(76) + 2(67) + 2(68) + 2(69) + 2(72) + 2(68) + 2(56) + 2(42) + 2(23) + 0]$

$= 7435$ sq m

46. The quadratic polynomial

$$p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}y_1 + \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}y_2 + \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}y_3$$

passes through the three points.