CHAPTER 1
Limits and Their Properties

C H A P T E R 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

Solutions to Odd-Numbered Exercises

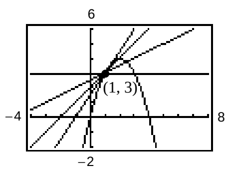
1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300$ feet

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$sq. units

7. Precalculus: Volume $= (2)(4)(3) = 24$ cubic units

9. (a)



(b) The graphs of y_2 are approximations to the tangent line to y_1 at $x = 1$.

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:

$\{0.2, 0.1, 0.01, 0.001\}$

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$\lim_{x\to 2} \frac{x-2}{x^2 - x - 2} \approx 0.3333$ (Actual limit is $\frac{1}{3}$.)

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$\lim_{x\to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x} \approx 0.2887$ (Actual limit is $1/(2\sqrt{3})$.)

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$$\lim_{x\to 3}\frac{[1/(x+1)]-(1/4)}{x-3}\approx -0.0625 \quad \left(\text{Actual limit is } -\tfrac{1}{16}.\right)$$

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$$\lim_{x\to 0}\frac{\sin x}{x}\approx 1.0000$$ (Actual limit is 1.) (Make sure you use radian mode.)

9. $\lim_{x\to 3}(4-x)=1$

11. $\lim_{x\to 2}f(x)=\lim_{x\to 2}(4-x)=2$

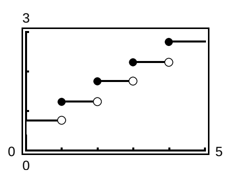
13. $\lim_{x\to 5}\frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1, whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1.

15. $\lim_{x\to\pi/2}\tan x$ does not exist since the function increases and decreases without bound as x approaches $\pi/2$.

17. $\lim_{x\to 0}\cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0.

19. $C(t)=0.75-0.50[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$\lim_{t\to 3.5}C(t)=2.25$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t\to 3}C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t=3$.

21. You need to find δ such that $0<|x-1|<\delta$ implies $|f(x)-1|=\left|\frac{1}{x}-1\right|<0.1$. That is,

$$-0.1<\frac{1}{x}-1<0.1$$
$$1-0.1<\frac{1}{x}<1+0.1$$
$$\frac{9}{10}<\frac{1}{x}<\frac{11}{10}$$
$$\frac{10}{9}>x>\frac{10}{11}$$
$$\frac{10}{9}-1>x-1>\frac{10}{11}-1$$
$$\frac{1}{9}>x-1>-\frac{1}{11}.$$

So take $\delta=\frac{1}{11}$. Then $0<|x-1|<\delta$ implies

$$-\frac{1}{11}<x-1<\frac{1}{11}$$
$$-\frac{1}{11}<x-1<\frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x)-1|=\left|\frac{1}{x}-1\right|<\epsilon<0.1.$$

23. $\lim_{x\to 2} (3x + 2) = 8 = L$

$$|(3x + 2) - 8| < 0.01$$
$$|3x - 6| < 0.01$$
$$3|x - 2| < 0.01$$
$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

Hence, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$
$$|3x - 6| < 0.01$$
$$|(3x + 2) - 8| < 0.01$$
$$|f(x) - L| < 0.01$$

25. $\lim_{x\to 2} (x^2 - 3) = 1 = L$

$$|(x^2 - 3) - 1| < 0.01$$
$$|x^2 - 4| < 0.01$$
$$|(x + 2)(x - 2)| < 0.01$$
$$|x + 2|\,|x - 2| < 0.01$$
$$|x - 2| < \frac{0.01}{|x + 2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$
$$|x + 2||x - 2| < 0.01$$
$$|x^2 - 4| < 0.01$$
$$|(x^2 - 3) - 1| < 0.01$$
$$|f(x) - L| < 0.01$$

27. $\lim_{x\to 2} (x + 3) = 5$

Given $\epsilon > 0$:

$$|(x + 3) - 5| < \epsilon$$
$$|x - 2| < \epsilon = \delta$$

Hence, let $\delta = \epsilon$.

Hence, if $0 < |x - 2| < \delta = \epsilon$, you have

$$|x - 2| < \epsilon$$
$$|(x + 3) - 5| < \epsilon$$
$$|f(x) - L| < \epsilon$$

29. $\lim_{x\to -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\epsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \epsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \epsilon$$
$$\tfrac{1}{2}|x - (-4)| < \epsilon$$
$$|x - (-4)| < 2\epsilon$$

Hence, let $\delta = 2\epsilon$.

Hence, if $0 < |x - (-4)| < \delta = 2\epsilon$, you have

$$|x - (-4)| < 2\epsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \epsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \epsilon$$
$$|f(x) - L| < \epsilon$$

31. $\lim_{x\to 6} 3 = 3$

Given $\epsilon > 0$:

$$|3 - 3| < \epsilon$$
$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|3 - 3| < \epsilon$$
$$|f(x) - L| < \epsilon$$

33. $\lim_{x\to 0} \sqrt[3]{x} = 0$

Given $\epsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \epsilon$

$$\left|\sqrt[3]{x}\right| < \epsilon$$
$$|x| < \epsilon^3 = \delta$$

Hence, let $\delta = \epsilon^3$.

Hence for $0 < |x - 0| < \delta = \epsilon^3$, you have

$$|x| < \epsilon^3$$
$$|\sqrt[3]{x}| < \epsilon$$
$$|\sqrt[3]{x} - 0| < \epsilon$$
$$|f(x) - L| < \epsilon$$

35. $\lim_{x\to-2} |x-2| = |(-2)-2| = 4$

Given $\epsilon > 0$:

$$||x-2|-4| < \epsilon$$
$$|-(x-2)-4| < \epsilon \quad (x-2<0)$$
$$|-x-2| = |x+2| = |x-(-2)| < \epsilon$$

Hence, $\delta = \epsilon$.

Hence for $0 < |x-(-2)| < \delta = \epsilon$, you have

$$|x+2| < \epsilon$$
$$|-(x+2)| < \epsilon$$
$$|-(x-2)-4| < \epsilon$$
$$||x-2|-4| < \epsilon \quad \text{(because } x-20\text{)}$$
$$|f(x)-L| < \epsilon$$

37. $\lim_{x\to1} (x^2+1) = 2$

Given $\epsilon > 0$:

$$|(x^2+1)-2| < \epsilon$$
$$|x^2-1| < \epsilon$$
$$|(x+1)(x-1)| < \epsilon$$
$$|x-1| < \frac{\epsilon}{|x+1|}$$

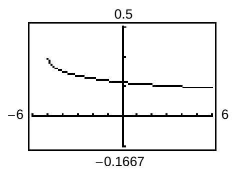
If we assume $0 < x < 2$, then $\delta = \epsilon/3$.

Hence for $0 < |x-1| < \delta = \frac{\epsilon}{3}$, you have

$$|x-1| < \frac{1}{3}\epsilon < \frac{1}{|x+1|}\epsilon$$
$$|x^2-1| < \epsilon$$
$$|(x^2+1)-2| < \epsilon$$
$$|f(x)-2| < \epsilon$$

39. $f(x) = \frac{\sqrt{x+5}-3}{x-4}$

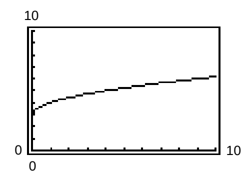
$\lim_{x\to4} f(x) = \frac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$.
The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

41. $f(x) = \frac{x-9}{\sqrt{x}-3}$

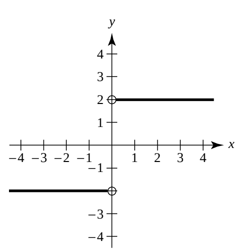
$\lim_{x\to9} f(x) = 6$



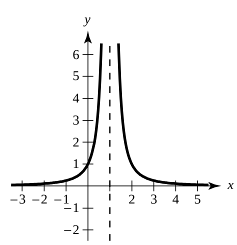
The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

43. $\lim_{x\to8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

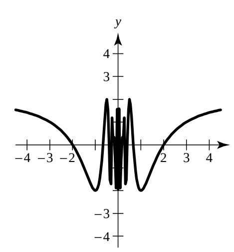
45. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:

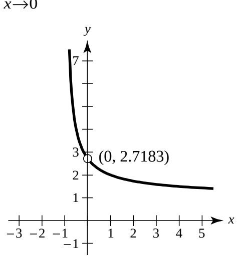


(iii) The values of f oscillate between two fixed numbers as x approaches c:



47. $f(x) = (1+x)^{1/x}$

$\lim_{x\to0} (1+x)^{1/x} = e \approx 2.71828$



x	$f(x)$
−0.1	2.867972
−0.01	2.731999
−0.001	2.719642
−0.0001	2.718418
−0.00001	2.718295
−0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

49. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$. From Exercise 7, we have

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

51. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$f(4) = 10$

$\lim_{x\to 4} f(x) = \lim_{x\to 4} (x^2 - 4x) = 0 \neq 10$

53. Answers will vary.

55. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, we have

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.$$

Therefore, $|L_1 - L_2| < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

57. $\lim_{x\to c} [f(x) - L] = 0$ means that for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \epsilon.$$

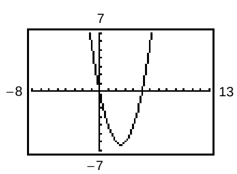
This means the same as $|f(x) - L| < \epsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x\to c} f(x) = L$.

Section 1.3 Evaluating Limits Analytically

1.

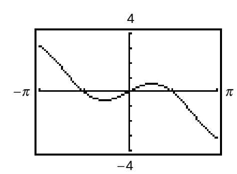


$h(x) = x^2 - 5x$

(a) $\lim_{x\to 5} h(x) = 0$

(b) $\lim_{x\to -1} h(x) = 6$

3.



$f(x) = x \cos x$

(a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to \pi/3} f(x) \approx 0.524$ $\left(= \frac{\pi}{6}\right)$

5. $\lim_{x\to 2} x^4 = 2^4 = 16$

7. $\lim_{x\to 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x\to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

11. $\lim_{x\to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

13. $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$

15. $\lim_{x\to 1} \frac{x - 3}{x^2 + 4} = \frac{1 - 3}{1^2 + 4} = \frac{-2}{5} = -\frac{2}{5}$

17. $\lim_{x\to 7} \frac{5x}{\sqrt{x + 2}} = \frac{5(7)}{\sqrt{7 + 2}} = \frac{35}{\sqrt{9}} = \frac{35}{3}$

19. $\lim_{x\to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

21. $\lim_{x\to -4}(x+3)^2 = (-4+3)^2 = 1$

23. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

25. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

27. $\lim_{x\to \pi/2} \sin x = \sin\frac{\pi}{2} = 1$

29. $\lim_{x\to 2} \cos\frac{\pi x}{3} = \cos\frac{\pi 2}{3} = -\frac{1}{2}$

31. $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

33. $\lim_{x\to 5\pi/6} \sin x = \sin\frac{5\pi}{6} = \frac{1}{2}$

35. $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right) = \tan\frac{3\pi}{4} = -1$

37. (a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c} g(x) = 5(3) = 15$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (2)(3) = 6$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{3}$

39. (a) $\lim_{x\to c}[f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}[3f(x)] = 3\lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}[f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

41. $f(x) = -2x + 1$ and $g(x) = \frac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 1$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 3$

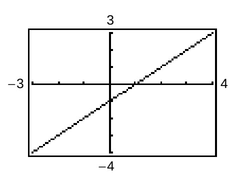
43. $f(x) = x(x+1)$ and $g(x) = \frac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 0$

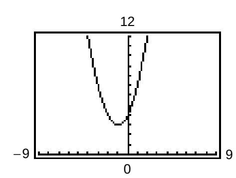
45. $f(x) = \frac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -2$



47. $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 12$



49. $\lim_{x\to 5}\frac{x - 5}{x^2 - 25} = \lim_{x\to 5}\frac{x - 5}{(x+5)(x-5)}$

$= \lim_{x\to 5}\frac{1}{x + 5} = \frac{1}{10}$

51. $\lim_{x\to -3}\frac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3}\frac{(x+3)(x-2)}{(x+3)(x-3)}$

$= \lim_{x\to -3}\frac{x - 2}{x - 3} = \frac{-5}{-6} = \frac{5}{6}$

53. $\lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x} \cdot \frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$$= \lim_{x\to 0} \frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+5}+\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

55. $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$$= \lim_{x\to 4} \frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)} = \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

57. $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{\frac{2-(2+x)}{2(2+x)}}{x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}$

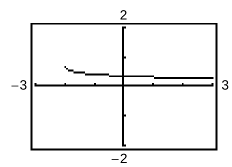
59. $\lim_{\Delta x\to 0} \frac{2(x+\Delta x)-2x}{\Delta x} = \lim_{\Delta x\to 0} \frac{2x+2\Delta x-2x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$

61. $\lim_{\Delta x\to 0} \frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$$= \lim_{\Delta x\to 0} (2x+\Delta x-2) = 2x-2$$

63. $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \approx 0.354$

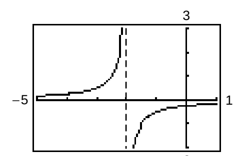
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.345	?	0.354	0.353	0.349



Analytically, $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \cdot \frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$= \lim_{x\to 0} \frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354$$

65. $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = -\frac{1}{4}$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238

Analytically, $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim_{x\to 0} \frac{2-(2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$

67. $\lim_{x\to 0} \frac{\sin x}{5x} = \lim_{x\to 0} \left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

69. $\lim_{x\to 0} \frac{\sin x(1-\cos x)}{2x^2} = \lim_{x\to 0}\left[\frac{1}{2}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$

$= \frac{1}{2}(1)(0) = 0$

71. $\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

73. $\lim_{h\to 0} \frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$

$= (0)(0) = 0$

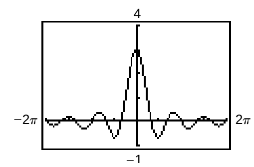
75. $\lim_{x\to \pi/2} \frac{\cos x}{\cot x} = \lim_{x\to \pi/2} \sin x = 1$

77. $\lim_{t\to 0} \frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

79. $f(t) = \frac{\sin 3t}{t}$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

Analytically, $\lim_{t\to 0} \frac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\frac{\sin 3t}{3t}\right) = 3(1) = 3.$

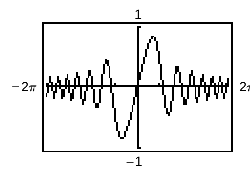


The limit appear to equal 3.

81. $f(x) = \frac{\sin x^2}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998

Analytically, $\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} x\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0.$



83. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)+3-(2x+3)}{h} = \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$

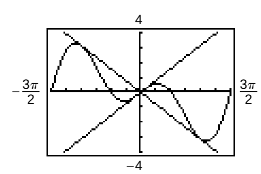
85. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} = \lim_{h\to 0} \frac{4x-4(x+h)}{(x+h)xh} = \lim_{h\to 0} \frac{-4}{(x+h)x} = \frac{-4}{x^2}$

87. $\lim_{x\to 0} (4-x^2) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0} (4+x^2)$

$4 \le \lim_{x\to 0} f(x) \le 4$

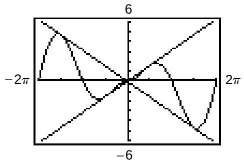
Therefore, $\lim_{x\to 0} f(x) = 4.$

89. $f(x) = x\cos x$



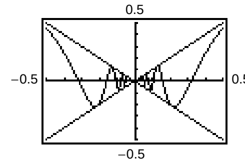
$\lim_{x\to 0} (x\cos x) = 0$

91. $f(x) = |x| \sin x$



$\lim_{x \to 0} |x| \sin x = 0$

93. $f(x) = x \sin \dfrac{1}{x}$



$\lim_{x \to 0} \left(x \sin \dfrac{1}{x} \right) = 0$

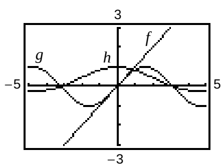
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

99. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When you are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. Thus, $|g|/|f| \approx 1$ when x is "close to" 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \to 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \to 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \to 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \to 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed = 160 ft/sec

103. $s(t) = -4.9t^2 + 150$

$$\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \to 3} \frac{-4.9(9 - t^2)}{3 - t}$$

$$= \lim_{x \to 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{x \to 3} -4.9(3 + t) = -29.4 \text{ m/sec}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist.

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[\frac{1}{x} + \left(-\frac{1}{x} \right) \right] = \lim_{x \to 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \epsilon$ for any $\epsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\epsilon > 0$ be given. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon$$

which implies that $\lim_{x \to c} [bf(x)] = bL$.

111. $-M|f(x)| \le f(x)g(x) \le M|f(x)|$

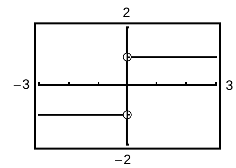
$$\lim_{x\to c}(-M|f(x)|) \le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}(M|f(x)|)$$

$$-M(0) \le \lim_{x\to c} f(x)g(x) \le M(0)$$

$$0 \le \lim_{x\to c} f(x)g(x) \le 0$$

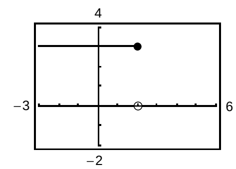
Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

113. False. As x approaches 0 from the left, $\dfrac{|x|}{x} = -1$.



115. True.

117. False. The limit does not exist.



119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \ge 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} 4 = 4.$$

$\lim_{x\to 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

121. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

$\lim_{x\to 0} g(x) = 0$.

When x is "close to" 0, both parts of the function are "close to" 0.

123. (a) $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$$

$$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(b) Thus, $\dfrac{1-\cos x}{x^2} \approx \dfrac{1}{2} \Rightarrow 1-\cos x \approx \dfrac{1}{2}x^2$

$\Rightarrow \cos x \approx 1 - \dfrac{1}{2}x^2$ for $x \approx 0$.

(c) $\cos(0.1) \approx 1 - \dfrac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x\to 3^+} f(x) = 1$

(b) $\lim_{x\to 3^-} f(x) = 1$

(c) $\lim_{x\to 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x\to 4^+} f(x) = 2$

(b) $\lim_{x\to 4^-} f(x) = -2$

(c) $\lim_{x\to 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x\to 5^+} \frac{x-5}{x^2-25} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$

9. $\lim_{x\to -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ grows without bound as $x \to -3^-$.

11. $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1.$

13.
$$\lim_{\Delta x\to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)}$$
$$= \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

15. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+2}{2} = \frac{5}{2}$

17. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^3+1) = 2$

$\lim_{x\to 1} f(x) = 2$

19. $\lim_{x\to \pi} \cot x$ does not exist since

$\lim_{x\to \pi^+} \cot x$ and $\lim_{x\to \pi^-} \cot x$ do not exist.

21. $\lim_{x\to 4^-} (3[\![x]\!] - 5) = 3(3) - 5 = 4$
($[\![x]\!] = 3$ for $3 < x < 4$)

23. $\lim_{x\to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x\to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x\to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

25. $f(x) = \frac{1}{x^2-4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \frac{[\![x]\!]}{2} + x$

has discontinuities at each integer k since $\lim_{x\to k^-} f(x) \neq \lim_{x\to k^+} f(x)$.

29. $g(x) = \sqrt{25-x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x\to 0^-} f(x) = 3 = \lim_{x\to 0^+} f(x)$.
f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x.

35. $f(x) = 3x - \cos x$ is continuous for all real x.

37. $f(x) = \dfrac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since

$\dfrac{x}{x^2 - x} = \dfrac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

39. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

41. $f(x) = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x\to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \dfrac{|x + 2|}{x + 2}$ has a nonremovable discontinuity at $x = -2$ since $\lim_{x\to -2} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim_{x\to 1} f(x) = 1$

3. $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

47. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$ has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(\dfrac{x}{2} + 1\right) = 2 \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3 - x) = 1 \end{array}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases} = \begin{cases} \tan \dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$ has **possible** discontinuities at $x = -1$, $x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim_{x\to -1} f(x) = -1$ $\qquad \lim_{x\to 1} f(x) = 1$

3. $f(-1) = \lim_{x\to -1} f(x)$ $\qquad f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

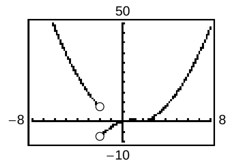
51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

53. $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer k.

55. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -2$.



57. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \implies a = \dfrac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)\ 3a + b &= -2 \\ \hline 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

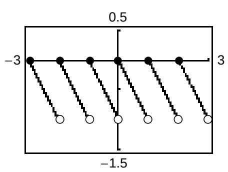
Continuous for all real x.

63. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

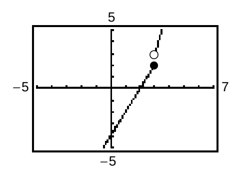
65. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



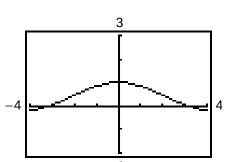
69. $f(x) = \dfrac{x}{x^2 + 1}$

Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \dfrac{\pi x}{4}$

Continuous on:
$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

73. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

77. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π.

79. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

81. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

83. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$$c = 3 \ (x = -4 \text{ is not in the interval.})$$

Thus, $f(3) = 11$.

85. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$(x^2 + x + 3$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

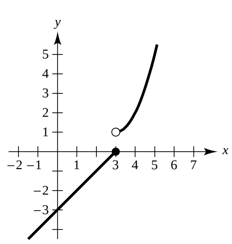
87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

89.



The function is not continuous at $x = 3$ because $\lim_{x\to3^+} f(x) = 1 \neq 0 = \lim_{x\to3^-} f(x)$.

91. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{ for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

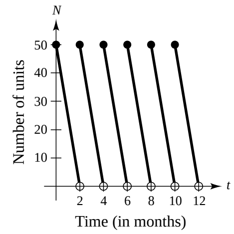
$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!]$.

For example, $f\left(\frac{1}{2}\right) = 3 + 0 = 3$, $g\left(\frac{1}{2}\right) = 3 - (-1) = 4$.

93. $N(t) = 25\left(2\left[\!\left[\frac{t+2}{2}\right]\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer.The company replenishes its inventory every two months.



95. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere of radius r.

$V(1) = \frac{4}{3}\pi \approx 4.19$

$V(5) = \frac{4}{3}\pi(5^3) \approx 523.6$

Since $4.19 < 275 < 523.6$, the Intermediate Value Theorem implies that there is at least one value r between 1 and 5 such that $V(r) = 275$. (In fact, $r \approx 4.0341$.)

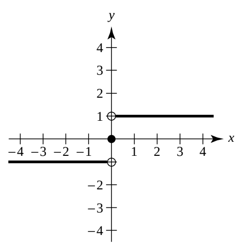
97. Let c be any real number. Then $\lim_{x\to c} f(x)$ does not exist since there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

99. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

(a) $\lim_{x\to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim_{x\to 0^+} \operatorname{sgn}(x) = 1$

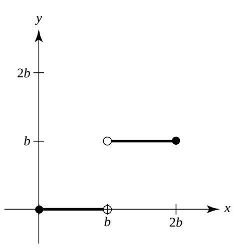
(c) $\lim_{x\to 0} \operatorname{sgn}(x)$ does not exist.



101. True; if $f(x) = g(x)$, $x \neq c$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ and at least one of these limits (if they exist) does not equal the corresponding function at $x = c$.

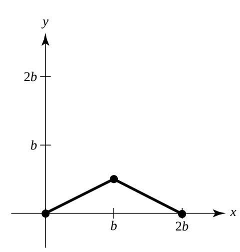
103. False; $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

105. (a) $f(x) = \begin{cases} 0 & 0 \le x < b \\ b & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2} & 0 \le x \le b \\ b - \dfrac{x}{2} & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

107. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, \; c > 0$

Domain: $x + c^2 \geq 0 \implies x \geq -c^2$ and $x \neq 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c}$$

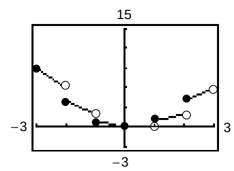
$$= \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

109. $h(x) = x[\![x]\!]$

h has nonremovable discontinuities at

$x = \pm 1, \pm 2, \pm 3, \ldots$.



Section 1.5 Infinite Limits

1. $\displaystyle\lim_{x\to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

$\displaystyle\lim_{x\to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

3. $\displaystyle\lim_{x\to -2^+} \tan \frac{\pi x}{4} = -\infty$

$\displaystyle\lim_{x\to -2^-} \tan \frac{\pi x}{4} = \infty$

5. $f(x) = \dfrac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$\displaystyle\lim_{x\to -3^-} f(x) = \infty$

$\displaystyle\lim_{x\to -3^+} f(x) = -\infty$

7. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$\displaystyle\lim_{x\to -3^-} f(x) = \infty$

$\displaystyle\lim_{x\to -3^+} f(x) = -\infty$

9. $\lim_{x\to 0^+} \frac{1}{x^2} = \infty = \lim_{x\to 0^-} \frac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

11. $\lim_{x\to 2^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$

$\lim_{x\to 2^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$

Therefore, $x = 2$ is a vertical asymptote.

$\lim_{x\to -1^+} \frac{x^2 - 2}{(x-2)(x+1)} = \infty$

$\lim_{x\to -1^-} \frac{x^2 - 2}{(x-2)(x+1)} = -\infty$

Therefore, $x = -1$ is a vertical asymptote.

13. $\lim_{x\to -2^-} \frac{x^2}{x^2 - 4} = \infty$ and $\lim_{x\to -2^+} \frac{x^2}{x^2 - 4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 2^-} \frac{x^2}{x^2 - 4} = -\infty$ and $\lim_{x\to 2^+} \frac{x^2}{x^2 - 4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

17. $f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x}$ has vertical asymptotes at

$$x = \frac{(2n+1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2},\ n \text{ any integer.}$$

19. $\lim_{t\to 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t\to 0^-} \left(1 - \frac{4}{t^2}\right)$

Therefore, $t = 0$ is a vertical asymptote.

21. $\lim_{x\to -2^+} \frac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to -2^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim_{x\to 1^+} \frac{x}{(x+2)(x-1)} = \infty$

$\lim_{x\to 1^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

23. $f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x+1)(x^2 - x + 1)}{x + 1}$

has no vertical asymptote since

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2 - x + 1) = 3$$

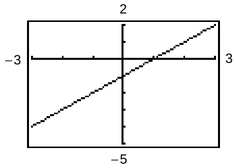
25. $f(x) = \frac{(x-5)(x+3)}{(x-5)(x^2+1)} = \frac{x+3}{x^2+1},\ x \neq 5$

No vertical asymptotes. The graph has a hole at $x = 5$.

27. $s(t) = \frac{t}{\sin t}$ has vertical asymptotes at $t = n\pi$, n a nonzero integer. There is no vertical asymptote at $t = 0$ since

$$\lim_{t\to 0} \frac{t}{\sin t} = 1.$$

29. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} (x - 1) = -2$

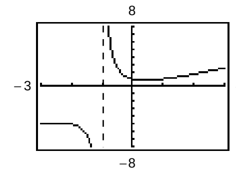


Removable discontinuity at $x = -1$

31. $\lim_{x \to -1^+} \frac{x^2 + 1}{x + 1} = \infty$

$\lim_{x \to -1^-} \frac{x^2 + 1}{x + 1} = -\infty$

Vertical asymptote at $x = -1$



33. $\lim_{x \to 2^+} \frac{x - 3}{x - 2} = -\infty$

35. $\lim_{x \to 3^+} \frac{x^2}{(x - 3)(x + 3)} = \infty$

37. $\lim_{x \to -3^-} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \to -3^-} \frac{x - 1}{x - 2} = \frac{4}{5}$

39. $\lim_{x \to 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x^2 + 1} = \frac{1}{2}$

41. $\lim_{x \to 0^-} \left(1 + \frac{1}{x}\right) = -\infty$

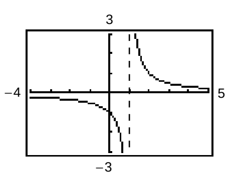
43. $\lim_{x \to 0^+} \frac{2}{\sin x} = \infty$

45. $\lim_{x \to \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \to \pi} \left(\sqrt{x} \sin x\right) = 0$

47. $\lim_{x \to (1/2)^-} x \sec(\pi x) = \infty$ and $\lim_{x \to (1/2)^+} x \sec(\pi x) = -\infty$.
Therefore, $\lim_{x \to (1/2)} x \sec(\pi x)$ does not exist.

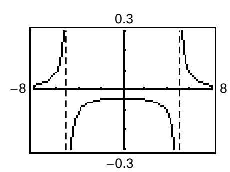
49. $f(x) = \frac{x^2 + x + 1}{x^3 - 1}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x - 1} = \infty$



51. $f(x) = \frac{1}{x^2 - 25}$

$\lim_{x \to 5^-} f(x) = -\infty$



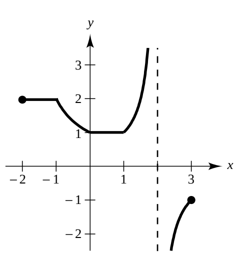
53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x \to c} f(x) = \infty$$

says how the limit fails to exist.

55. One answer is $f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12}$.

57.



59. $S = \frac{k}{1 - r}$, $0 < |r| < 1$. Assume $k \neq 0$.

$\lim_{r \to 1^-} S = \lim_{r \to 1^-} \frac{k}{1 - r} = \infty \quad (\text{or } -\infty \text{ if } k < 0)$

61. $C = \dfrac{528x}{100 - x}, \; 0 \le x < 100$

(a) $C(25) = \$176$ million

(b) $C(50) = \$528$ million

(c) $C(75) = \$1584$ million

(d) $\displaystyle\lim_{x\to 100^-} \frac{528}{100 - x} = \infty$ Thus, it is not possible.

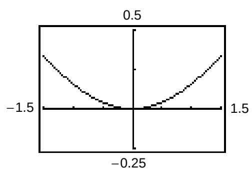
63. (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\displaystyle\lim_{x\to 25^-} \frac{2x}{\sqrt{625 - x^2}} = \infty$

65. (a)

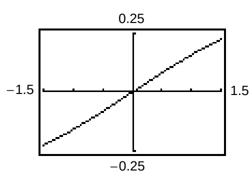
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x} = 0$

(b)

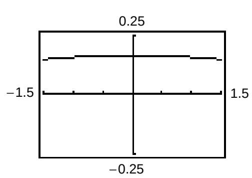
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^2} = 0$

(c)

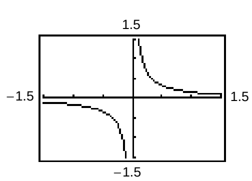
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^3} = 0.1167 \;\; (1/6)$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^4} = \infty$

For $n \ge 3$, $\displaystyle\lim_{x\to 0^+} \frac{x - \sin x}{x^n} = \infty$.

67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20 \cot \phi) + 2(10 \cot \phi)$: straight sections. The angle subtended in each circle is

$$2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.$$

Thus, the length of the belt around the pulleys is

$$20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).$$

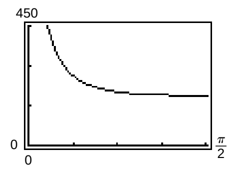
Total length $= 60 \cot \phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\lim_{\phi \to (\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi \to 0^+} L = \infty$

69. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ or}$$

$$g(x) = \frac{x}{x^2 + 1}.$$

71. False; let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

73. Given $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$:

(2) Product:

If $L > 0$, then for $\epsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_1$. Thus, $L/2 < g(x) < 3L/2$. Since $\lim_{x \to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $|x - c| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim_{x \to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\epsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\epsilon$ whenever $0 < |x - c| < \delta_1$ and there exists $\delta_2 > 0$ such that $|g(x) - L| < L/2$ whenever $0 < |x - c| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < |x - c| < \delta$, we have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.$$

Therefore, $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$.

75. Given $\lim_{x \to c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \to c} f(x)$ exists and equals L. Then,

$$\lim_{x \to c} \frac{1}{f(x)} = \frac{\lim_{x \to c} 1}{\lim_{x \to c} f(x)} = \frac{1}{L} = 0.$$

This is not possible. Thus, $\lim_{x \to c} f(x)$ does not exist.