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CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus
Solutions to Odd-Numbered Exercises

1. Precalculus: (20 ft/sec)(15 seconds) = 300 feet

3. Calculus required: slope of tangent line at x = 2 is rate of change, and equals about 0.16.

5. Precalculus: Area = \( \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2} \) sq. units

7. Precalculus: Volume = (2)(4)(3) = 24 cubic units

9. (a)

(b) The graphs of \( y_2 \) are approximations to the tangent line to \( y_1 \) at \( x = 1 \).

(c) The slope is approximately 2. For a better approximation make the list numbers smaller:
\{0.2, 0.1, 0.01, 0.001\}

11. (a) \( D_1 = \sqrt{(5 - 1)^2 + (1 - 5)^2} = \sqrt{16 + 16} = 5.66 \)

(b) \( D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{3}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{1}{2}\right)^2} + \sqrt{1 + \left(\frac{1}{2} - 1\right)^2} \approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11 \)

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

<table>
<thead>
<tr>
<th>x</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
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<tbody>
<tr>
<td>f(x)</td>
<td>0.3448</td>
<td>0.3344</td>
<td>0.3334</td>
<td>0.3332</td>
<td>0.3322</td>
<td>0.3226</td>
</tr>
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</table>

\[ \lim_{x \to 2} \frac{x - 2}{x^2 - x - 2} = 0.3333 \] (Actual limit is \( \frac{1}{2} \))

3.

<table>
<thead>
<tr>
<th>x</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>0.2911</td>
<td>0.289</td>
<td>0.287</td>
<td>0.2887</td>
<td>0.2884</td>
<td>0.2863</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x} = 0.2887 \] (Actual limit is \( 1/(2\sqrt{3}) \))
5. \[ f(x) = \begin{array}{cccccc} x & 2.9 & 2.99 & 2.999 & 3.001 & 3.01 & 3.1 \\ \hline f(x) & -0.0641 & -0.0627 & -0.0625 & -0.0625 & -0.0623 & -0.0610 \\
\end{array} \]

\[ \lim_{x \to 3} \frac{1/(x+1) - (1/4)}{x-3} = -0.0625 \quad \text{(Actual limit is \(-\frac{1}{11}\))} \]

7. \[ f(x) = \begin{array}{cccccc} x & -0.1 & -0.01 & -0.001 & 0.001 & 0.1 \\ \hline f(x) & 0.9983 & 0.99998 & 1.0000 & 1.0000 & 0.99998 & 0.9983 \\
\end{array} \]

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1.0000 \quad \text{(Actual limit is 1.) (Make sure you use radian mode.)} \]

9. \[ \lim_{x \to 3} (4 - x) = 1 \]

11. \[ \lim_{x \to 2} f(x) = \lim_{x \to 2} (4 - x) = 2 \]

13. \[ \lim_{x \to 5} \frac{|x - 5|}{x - 5} \]

does not exist. For values of \(x\) to the left of 5, \(\frac{|x - 5|}{x - 5}\) equals \(-1\), whereas for values of \(x\) to the right of 5, \(\frac{|x - 5|}{x - 5}\) equals 1.

15. \[ \lim_{x \to \pi/2} \tan x \]
does not exist since the function increases and decreases without bound as \(x\) approaches \(\pi/2\).

17. \[ \lim_{x \to 0} \cos\left(\frac{1}{x}\right) \]
does not exist since the function oscillates between \(-1\) and 1 as \(x\) approaches 0.

19. \[ C(t) = 0.75 - 0.50[\{(t - 1)\}] \]

(b) \[ C \begin{array}{cccccc} t & 3 & 3.3 & 3.4 & 3.5 & 3.6 & 3.7 & 4 \\ \hline C & 1.75 & 2.25 & 2.25 & 2.25 & 2.25 & 2.25 & 2.25 \\
\end{array} \]

\[ \lim_{t \to 3.5} C(t) = 2.25 \]

(c) \[ C \begin{array}{cccccc} t & 2 & 2.5 & 2.9 & 3 & 3.1 & 3.5 & 4 \\ \hline C & 1.25 & 1.75 & 1.75 & 2.25 & 2.25 & 2.25 & 2.25 \\
\end{array} \]

\[ \lim_{t \to 3} C(t) \]
does not exist. The values of \(C\) jump from 1.75 to 2.25 at \(t = 3\).

21. You need to find \(\delta\) such that \(0 < |x - 1| < \delta\) implies \(|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < 0.1\). That is,

\[-0.1 < \frac{1}{x} - 1 < 0.1 \]

\[1 - 0.1 < \frac{1}{x} < 1 + 0.1 \]

\[\frac{9}{10} < \frac{1}{x} < \frac{11}{10} \]

\[\frac{10}{9} > x > \frac{10}{11} \]

\[\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1 \]

\[\frac{1}{9} > x - 1 > -\frac{1}{11} \]

So take \(\delta = \frac{1}{11}\). Then \(0 < |x - 1| < \delta\) implies

\[-\frac{1}{11} < x - 1 < \frac{1}{11} \]

\[-\frac{1}{11} < x - 1 < \frac{1}{9} \]

Using the first series of equivalent inequalities, you obtain

\[|f(x) - 1| = \left| \frac{1}{x} - 1 \right| < \varepsilon < 0.1.\]
23. \(\lim_{x \to 2} (3x + 2) = 8 = L\)
\(|(3x + 2) - 8| < 0.01\)
\(|3x - 6| < 0.01\)
\(3|x - 2| < 0.01\)
\(0 < |x - 2| < \frac{0.01}{3} = 0.0033 = \delta\)
Hence, if \(0 < |x - 2| < \delta\), you have
\(3|x - 2| < 0.01\)
\(|3x - 6| < 0.01\)
\(|(3x + 2) - 8| < 0.01\)
\(|f(x) - L| < 0.01\)

25. \(\lim_{x \to 2} (x^2 - 3) = 1 = L\)
\(|(x^2 - 3) - 1| < 0.01\)
\(|x^2 - 4| < 0.01\)
\(|(x + 2)(x - 2)| < 0.01\)
\(|x + 2||x - 2| < 0.01\)
\(|x + 2| < 0.01\)
\(|x^2 - 4| < 0.01\)
\(|(x^2 - 3) - 1| < 0.01\)
\(|f(x) - L| < 0.01\)

If we assume \(1 < x < 3\), then \(\delta = 0.01/5 = 0.002\).
Hence, if \(0 < |x - 2| < \delta\), you have
\(|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)\)
\(|x + 2||x - 2| < 0.01\)
\(|x^2 - 4| < 0.01\)
\(|(x^2 - 3) - 1| < 0.01\)
\(|f(x) - L| < 0.01\)

27. \(\lim_{x \to 2} (x + 3) = 5\)
Given \(\varepsilon > 0:\)
\(|(x + 3) - 5| < \varepsilon\)
\(|x - 2| < \varepsilon = \delta\)
Hence, let \(\delta = \varepsilon\).
Hence, if \(0 < |x - 2| < \delta = \varepsilon\), you have
\(|x - 2| < \varepsilon\)
\(|(x + 3) - 5| < \varepsilon\)
\(|f(x) - L| < \varepsilon\)

29. \(\lim_{x \to -4} \left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3\)
Given \(\varepsilon > 0:\)
\(|\left(\frac{1}{2}x - 1\right) - (-3)| < \varepsilon\)
\(|\frac{1}{2}x + 2| < \varepsilon\)
\(\left|\frac{1}{2}x - (-4)\right| < \varepsilon\)
\(|x - (-4)| < 2\varepsilon\)
Hence, let \(\delta = 2\varepsilon\).
Hence, if \(0 < |x - (-4)| < \delta = 2\varepsilon\), you have
\(|x - (-4)| < 2\varepsilon\)
\(|\frac{1}{2}x + 2| < \varepsilon\)
\(|\left(\frac{1}{2}x - 1\right) + 3| < \varepsilon\)
\(|f(x) - L| < \varepsilon\)

31. \(\lim_{x \to 6} 3 = 3\)
Given \(\varepsilon > 0:\)
\(|3 - 3| < \varepsilon\)
\(0 < \varepsilon\)
Hence, any \(\delta > 0\) will work.
Hence, for any \(\delta > 0\), you have
\(|3 - 3| < \varepsilon\)
\(|f(x) - L| < \varepsilon\)

33. \(\lim_{x \to 0} \sqrt{x} = 0\)
Given \(\varepsilon > 0:\)
\(|\sqrt{x} - 0| < \varepsilon\)
\(|\sqrt{x}| < \varepsilon\)
\(|x| < \varepsilon^3 = \delta\)
Hence, let \(\delta = \varepsilon^3\).
Hence for \(0 < |x - 0| < \delta = \varepsilon^3\), you have
\(|x| < \varepsilon^3\)
\(|\sqrt{x}| < \varepsilon\)
\(|\sqrt{x} - 0| < \varepsilon\)
\(|f(x) - L| < \varepsilon\)
35. \[ \lim_{x \to -2} |x - 2| = |(-2) - 2| = 4 \]
   Given \( \varepsilon > 0 \):
   \[ ||x - 2| - 4| < \varepsilon \]
   \[ |-(x - 2) - 4| < \varepsilon \quad (x - 2 < 0) \]
   \[ |-(x + 2)| < \varepsilon \]
   \[ |-(x - 2) - 4| < \varepsilon \quad (because x - 20) \]
   \[ |f(x) - L| < \varepsilon \]
   Hence, \( \delta = \varepsilon \).
   Hence for \( 0 < |x - (-2)| < \delta \) you have
   \[ |x + 2| < \varepsilon \]
   \[ |-(x + 2)| < \varepsilon \]
   \[ |-(x - 2) - 4| < \varepsilon \]
   \[ ||x - 2| - 4| < \varepsilon \quad (because x - 20) \]
   \[ |f(x) - L| < \varepsilon \]

39. \( f(x) = \frac{\sqrt{x + 5} - 3}{x - 4} \)
\[ \lim_{x \to 4} f(x) = \frac{1}{6} \]

41. \( f(x) = \frac{x - 9}{\sqrt{x} - 3} \)
\[ \lim_{x \to 9} f(x) = 6 \]

The domain is \([-5, 4) \cup (4, \infty)\).
The graphing utility does not show the hole at \((4, \frac{1}{6})\).

43. \( \lim_{x \to 8} f(x) = 25 \) means that the values of \( f \) approach 25 as \( x \) gets closer and closer to 8.

45. (i) The values of \( f \) approach different numbers as \( x \) approaches \( c \) from different sides of \( c \):

(ii) The values of \( f \) increase without bound as \( x \) approaches \( c \):

(iii) The values of \( f \) oscillate between two fixed numbers as \( x \) approaches \( c \):

47. \( f(x) = (1 + x)^{1/x} \)
\[ \lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828 \]

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<tr>
<td>-0.1</td>
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<th>( f(x) )</th>
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<tbody>
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<tr>
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<tr>
<td>0.000001</td>
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</table>
49. False; \( f(x) = \frac{\sin x}{x} \) is undefined when \( x = 0 \).
   From Exercise 7, we have
   \[
   \lim_{{x \to 0}} \frac{\sin x}{x} = 1.
   \]
   \[
   f(x) = \begin{cases} 
   x^2 - 4x, & x \neq 4 \\
   10, & x = 4 
   \end{cases}
   \]
   \[
   f(4) = 10
   \]
   \[
   \lim_{{x \to 4}} (x^2 - 4x) = 0 \neq 10
   \]

51. False; let
   \[
   \lim_{{x \to \pi/2}} \sin x = 1
   \]
   \[
   3 \leq \sin 2 \leq 10
   \]
   \[
   53. Answers will vary.
   \]

55. If \( \lim_{{x \to a}} f(x) = L_1 \) and \( \lim_{{x \to a}} f(x) = L_2 \), then for every \( \epsilon > 0 \), there exists \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that \( |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \epsilon \) and \( |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \epsilon \). Let \( \delta \) equal the smaller of \( \delta_1 \) and \( \delta_2 \). Then for \( |x - c| < \delta \), we have
   \[
   |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \epsilon + \epsilon.
   \]

Therefore, \( |L_1 - L_2| < 2\epsilon \). Since \( \epsilon > 0 \) is arbitrary, it follows that \( L_1 = L_2 \).

57. \( \lim_{{x \to c}} f(x) = L \) means that for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if
   \[
   0 < |x - c| < \delta,
   \]
   then
   \[
   |(f(x) - L) - 0| < \epsilon.
   \]
   This means the same as \( |f(x) - L| < \epsilon \) when
   \[
   0 < |x - c| < \delta.
   \]
   Thus, \( \lim_{{x \to c}} f(x) = L \).

Section 1.3 Evaluating Limits Analytically

1. \( h(x) = x^2 - 5x \)
   \( a ) \lim_{{x \to 5}} h(x) = 0 \)
   \( b ) \lim_{{x \to -1}} h(x) = 6 \)

3. \( f(x) = x \cos x \)
   \( a ) \lim_{{x \to 0}} f(x) = 0 \)
   \( b ) \lim_{{x \to -\pi/3}} f(x) \approx 0.524 \)

5. \( \lim_{{x \to 2}} x^2 = 4^2 = 16 \)

7. \( \lim_{{x \to 0}} (2x - 1) = 2(0) - 1 = -1 \)

9. \( \lim_{{x \to -3}} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0 \)

11. \( \lim_{{x \to -3}} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7 \)

13. \( \lim_{{x \to 2}} \frac{1}{x} = \frac{1}{2} \)

15. \( \lim_{{x \to 1}} \frac{x - 3}{x^2 + 4} = \frac{1 - 3}{1^2 + 4} = \frac{-2}{5} = -\frac{2}{5} \)

17. \( \lim_{{x \to 7}} \frac{5x}{\sqrt{x} + 2} = \frac{5(7)}{\sqrt{7} + 2} = \frac{35}{\sqrt{7} + 2} = \frac{35}{3} \)

19. \( \lim_{{x \to 3}} \sqrt{x + 1} = \sqrt{3 + 1} = 2 \)
21. \( \lim_{x \to -4} (x + 3)^2 = (-4 + 3)^2 = 1 \)

23. (a) \( \lim_{x \to 1} f(x) = 5 - 1 = 4 \)
   (b) \( \lim_{x \to -4} g(x) = 4^3 = 64 \)
   (c) \( \lim_{x \to 1} g(f(x)) = g(f(1)) = g(4) = 64 \)

25. (a) \( \lim_{x \to 4} f(x) = 4 - 1 = 3 \)
   (b) \( \lim_{x \to 3} g(x) = \sqrt{3 + 1} = 2 \)
   (c) \( \lim_{x \to 1} g(f(x)) = g(3) = 2 \)

27. \( \lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} = 1 \)

29. \( \lim_{x \to 2} \cos \frac{\pi x}{3} = \cos \frac{\pi \cdot 2}{3} = -\frac{1}{2} \)

33. \( \lim_{x \to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2} \)

37. (a) \( \lim_{x \to -3} [5g(x)] = 5 \lim_{x \to -3} g(x) = 5(3) = 15 \)
   (b) \( \lim_{x \to -3} [f(x) + g(x)] = \lim_{x \to -3} f(x) + \lim_{x \to -3} g(x) = 2 + 3 = 5 \)
   (c) \( \lim_{x \to -3} [f(x)g(x)] = \lim_{x \to -3} f(x) \lim_{x \to -3} g(x) = (2)(3) = 6 \)
   (d) \( \lim_{x \to -3} \frac{f(x)}{g(x)} = \lim_{x \to -3} \frac{f(x)}{\lim_{x \to -3} g(x)} = \frac{2}{3} \)

39. (a) \( \lim_{x \to 3} [f(x)]^3 = \left[ \lim_{x \to 3} f(x) \right]^3 = (4)^3 = 64 \)
   (b) \( \lim_{x \to 3} \sqrt{f(x)} = \sqrt{\lim_{x \to 3} f(x)} = \sqrt{4} = 2 \)
   (c) \( \lim_{x \to 3} [3f(x)] = 3 \lim_{x \to 3} f(x) = 3(4) = 12 \)
   (d) \( \lim_{x \to 3} [f(x)]^{1/2} = \left[ \lim_{x \to 3} f(x) \right]^{1/2} = (4)^{1/2} = 2 \)

41. \( f(x) = -2x + 1 \) and \( g(x) = \frac{-2x^2 + x}{x} \) agree except at \( x = 0 \).
   (a) \( \lim_{x \to 0} g(x) = \lim_{x \to 0} f(x) = 1 \)
   (b) \( \lim_{x \to -1} g(x) = \lim_{x \to -1} f(x) = 3 \)

45. \( f(x) = \frac{x^2 - 1}{x + 1} \) and \( g(x) = x - 1 \) agree except at \( x = -1 \).
   \[ \lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = -2 \]

49. \( \lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{x - 5}{(x + 5)(x - 5)} = \lim_{x \to 5} \frac{1}{x + 5} = \frac{1}{10} \)

51. \( \lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x \to -3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} = \lim_{x \to -3} \frac{x - 2}{x - 3} = \frac{-5}{6} = \frac{5}{6} \)
53. \[
\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} = \lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x + 5} + \sqrt{5}}{\sqrt{x + 5} + \sqrt{5}} = \lim_{x \to 0} \frac{(x + 5) - 5}{x(x + 5 + 5)} = \lim_{x \to 0} \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}
\]

55. \[
\lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4} \cdot \frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3} = \lim_{x \to 4} \frac{(x + 5) - 9}{(x - 4)(\sqrt{x + 5} + 3)} = \lim_{x \to 4} \frac{1}{\sqrt{x + 5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}
\]

57. \[
\lim_{x \to 0} \frac{1}{2 + x} - \frac{1}{2} = \lim_{x \to 0} \frac{2 - (2 + x)}{2(2 + x)} = \lim_{x \to 0} \frac{-1}{2(2 + x)} = -\frac{1}{4}
\]

59. \[
\lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2
\]

61. \[
\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2
\]

63. \[
\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x} = 0.354
\]

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<td>?</td>
<td>0.354</td>
<td>0.353</td>
<td>0.349</td>
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Analytically, \[
\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x + 2} + \sqrt{2}}{\sqrt{x + 2} + \sqrt{2}} = \lim_{x \to 0} \frac{x + 2 - 2}{x(x + 2 + \sqrt{2})} = \lim_{x \to 0} \frac{1}{\sqrt{x + 2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354
\]

65. \[
\lim_{x \to 0} \frac{1}{2 + x} - \frac{1}{2} = -\frac{1}{4}
\]

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<tr>
<td>f(x)</td>
<td>-0.263</td>
<td>-0.251</td>
<td>-0.250</td>
<td>?</td>
<td>-0.250</td>
<td>-0.249</td>
<td>-0.238</td>
</tr>
</tbody>
</table>

Analytically, \[
\lim_{x \to 0} \frac{1}{2 + x} - \frac{1}{2} = \lim_{x \to 0} \frac{2 - (2 + x)}{2(2 + x)} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-x}{2(2 + x)} \cdot \frac{1}{x} = \lim_{x \to 0} \frac{-1}{2(2 + x)} = -\frac{1}{4}
\]
67. \[ \lim_{x \to 0} \frac{\sin x}{5x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \frac{1}{5} \right) = \left( \frac{1}{5} \right) = \frac{1}{5} \]

69. \[ \lim_{x \to 0} \frac{\sin(x(1 - \cos x))}{2x^2} = \lim_{x \to 0} \left( \frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x} \right) = \frac{1}{2}(1)(0) = 0 \]

71. \[ \lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \cdot \sin x \right) = (1) \sin 0 = 0 \]

73. \[ \lim_{h \to 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \to 0} \frac{1 - \cos h}{h} \cdot (1 - \cos h) = (0)(0) = 0 \]

75. \[ \lim_{x \to \pi/2} \frac{\cos x}{\cot x} = \lim_{x \to \pi/2} \sin x = 1 \]

77. \[ \lim_{t \to 0} \frac{\sin 3t}{2t} = \lim_{t \to 0} \left( \frac{\sin 3t}{3t} \right) \left( \frac{3}{2} \right) = (1) \left( \frac{3}{2} \right) = \frac{3}{2} \]

79. \[ f(t) = \frac{\sin 3t}{t} \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>2.96</td>
<td>2.9996</td>
<td>3</td>
<td>?</td>
<td>3</td>
<td>2.9996</td>
<td>2.96</td>
</tr>
</tbody>
</table>

Analytically, \( \lim_{t \to 0} \frac{\sin 3t}{t} = \lim_{t \to 0} \left( \frac{\sin 3t}{3t} \right) = 3(1) = 3 \).

81. \[ f(x) = \frac{\sin^2 x}{x} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-0.099998</td>
<td>-0.01</td>
<td>-0.001</td>
<td>?</td>
<td>0.001</td>
<td>0.01</td>
<td>0.099998</td>
</tr>
</tbody>
</table>

Analytically, \( \lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \left( \frac{\sin x}{x} \right) = 0(1) = 0 \).

83. \[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x + h) + 3 - (2x + 3)}{h} = \lim_{h \to 0} \frac{2x + 2h + 3 - 2x - 3}{h} = \lim_{h \to 0} \frac{2h}{h} = 2 \]

85. \[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4 + \frac{4}{x + h} - 4}{h} = \lim_{h \to 0} \frac{4x - 4(x + h)}{(x + h)h} = \lim_{h \to 0} \frac{-4}{(x + h)x} = \frac{-4}{x^2} \]

87. \( \lim_{x \to 0} (4 - x^2) \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} (4 + x^2) \)

\[ 4 \leq \lim_{x \to 0} f(x) \leq 4 \]

Therefore, \( \lim_{x \to 0} f(x) = 4 \).

89. \[ f(x) = x \cos x \]

\[ \lim_{x \to 0} (x \cos x) = 0 \]
91. \( f(x) = |x| \sin x \)

\[
\lim_{x \to 0} |x| \sin x = 0
\]

93. \( f(x) = x \sin \frac{1}{x} \)

\[
\lim_{x \to 0} \left( x \sin \frac{1}{x} \right) = 0
\]

95. We say that two functions \( f \) and \( g \) agree at all but one point (on an open interval) if \( f(x) = g(x) \) for all \( x \) in the interval except for \( x = c \), where \( c \) is in the interval.

97. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as \( 0/0 \). That is,

\[
\lim_{x \to c} \frac{f(x)}{g(x)}
\]

for which \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \)

99. \( f(x) = x \), \( g(x) = \sin x \), \( h(x) = \frac{\sin x}{x} \)

When you are “close to” \( 0 \) the magnitude of \( f \) is approximately equal to the magnitude of \( g \).

Thus, \( \frac{|g|}{|f|} = 1 \) when \( x \) is “close to” \( 0 \).

101. \( s(t) = -16t^2 + 1000 \)

\[
\lim_{t \to 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \to 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \to 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \to 5} -16(t + 5) = -160 \text{ ft/sec.}
\]

Speed = 160 ft/sec

103. \( s(t) = -4.9t^2 + 150 \)

\[
\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \to 3} \frac{-4.9(9 - t^2)}{3 - t}
\]

\[
= \lim_{t \to 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{t \to 3} -4.9(3 + t) = -29.4 \text{ m/sec}
\]

105. Let \( f(x) = 1/x \) and \( g(x) = -1/x \). \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) do not exist.

\[
\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[ \frac{1}{x} + \left( -\frac{1}{x} \right) \right] = \lim_{x \to 0} [0] = 0
\]

107. Given \( f(x) = b \), show that for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) - b| < \epsilon \) whenever \( |x - c| < \delta \). Since \( |f(x) - b| = |b - b| = 0 < \epsilon \) for any \( \epsilon > 0 \), then any value of \( \delta > 0 \) will work.

109. If \( b = 0 \), then the property is true because both sides are equal to 0. If \( b \neq 0 \), let \( \epsilon > 0 \) be given. Since \( \lim_{x \to c} f(x) = L \), there exists \( \delta > 0 \) such that \( |f(x) - L| < \epsilon/|b| \) whenever \( 0 < |x - c| < \delta \). Hence, wherever \( 0 < |x - c| < \delta \), we have

\[
|b||f(x) - L| < \epsilon \quad \text{or} \quad |bf(x) - bL| < \epsilon
\]

which implies that \( \lim_{x \to c} [bf(x)] = bL \).
111. \[ M|f(x)| \leq f(x)g(x) \leq M|f(x)| \]
\[
\lim_{x \to c} (-M|f(x)|) \leq \lim_{x \to c} f(x)g(x) \leq \lim_{x \to c} (M|f(x)|)
\]
\[-M(0) \leq \lim_{x \to c} f(x)g(x) \leq M(0)\]
\[0 \leq \lim_{x \to c} f(x)g(x) \leq 0\]
Therefore, \( \lim_{x \to c} f(x)g(x) = 0 \).

115. True.

117. False. The limit does not exist.

119. Let
\[
f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}
\]
\[\lim_{x \to 0} |f(x)| = \lim_{x \to 0} 4 = 4.
\]
\[\lim_{x \to 0} f(x)\] does not exist since for \( x < 0 \), \( f(x) = -4 \) and for \( x \geq 0 \), \( f(x) = 4 \).

121. \( f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases} \)
\[g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases} \]
\[\lim_{x \to 0} f(x)\] does not exist.
No matter how “close to” 0 \( x \) is, there are still an infinite number of rational and irrational numbers so that \( \lim_{x \to 0} f(x) \) does not exist.
\[\lim_{x \to 0} g(x) = 0.\]
When \( x \) is “close to” 0, both parts of the function are “close to” 0.

123. (a) \[\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)} = \lim_{x \to 0} \sin^2 x \cdot \frac{1}{1 + \cos x} = (1) \left( \frac{1}{2} \right) = \frac{1}{2}\]
(b) Thus, \( \frac{1 - \cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x \approx \frac{1}{2} x^2 \)
\[\Rightarrow \cos x \approx 1 - \frac{1}{2} x^2 \text{ for } x = 0.\]
(c) \( \cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995\)
(d) \( \cos(0.1) \approx 0.9950, \) which agrees with part (c).
Section 1.4  Continuity and One-Sided Limits

1. (a) \( \lim_{x \to 3} f(x) = 1 \)
   (b) \( \lim_{x \to 3} f(x) = 1 \)
   (c) \( \lim_{x \to 3} f(x) = 1 \)

   The function is continuous at \( x = 3 \).

3. (a) \( \lim_{x \to 0} f(x) = 0 \)
   (b) \( \lim_{x \to 0} f(x) = 0 \)
   (c) \( \lim_{x \to 0} f(x) = 0 \)

   The function is NOT continuous at \( x = 3 \).

5. (a) \( \lim_{x \to 3} f(x) = 2 \)
   (b) \( \lim_{x \to 4} f(x) = -2 \)
   (c) \( \lim_{x \to 4} f(x) \) does not exist

   The function is NOT continuous at \( x = 4 \).

7. \( \lim_{x \to 3} \frac{x - 5}{x^2 - 25} = \lim_{x \to 3} \frac{1}{x + 5} = \frac{1}{10} \)

9. \( \lim_{x \to -3} \frac{x}{\sqrt{x^2 - 9}} \) does not exist because \( \frac{x}{\sqrt{x^2 - 9}} \) grows without bound as \( x \to -3^- \).

11. \( \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0} \frac{-x}{x} = -1 \).

13. \( \frac{1}{\Delta x} \frac{x + \Delta x - x}{x(x + \Delta x)} \), \( \lim_{\Delta x \to 0} -\Delta x \frac{-1}{x(x + 0)} = -\frac{1}{x^2} \)

15. \( \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x + 2}{2} = \frac{5}{2} \)

17. \( \lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 2 \)
   \( \lim_{x \to 1} f(x) = \lim_{x \to 1} (x^3 + 1) = 2 \)
   \( \lim_{x \to 1} f(x) = 2 \)

19. \( \lim_{x \to \pi} \cot x \) does not exist since \( \lim_{x \to \pi^-} \cot x \) and \( \lim_{x \to \pi^+} \cot x \) do not exist.

21. \( \lim_{x \to 3} (3[x] - 5) = 3(3) - 5 = 4 \)
   \( \lim_{x \to 4} (3[x] - 5) = 3(4) - 5 = 7 \)
   \( \lim_{x \to 0} (3[x] - 5) = 3(0) - 5 = -5 \)

23. \( \lim_{x \to 3} (2 - \lfloor -x \rfloor) \) does not exist because \( \lim_{x \to 3} (2 - \lfloor -x \rfloor) = 2 - (-3) = 5 \)
   and \( \lim_{x \to 3} (2 - \lfloor -x \rfloor) = 2 - (-4) = 6 \).

25. \( f(x) = \frac{1}{x^2 - 4} \) has discontinuities at \( x = -2 \) and \( x = 2 \) since \( f(-2) \) and \( f(2) \) are not defined.

27. \( f(x) = \frac{\|x\|}{2} + x \) has discontinuities at each integer \( k \) since \( \lim_{x \to k^+} f(x) \neq \lim_{x \to k^-} f(x) \).

29. \( g(x) = \sqrt{25 - x^2} \) is continuous on \([-5, 5]\).

31. \( \lim_{x \to 0} f(x) = 3 = \lim_{x \to 0} f(x) \)
   \( f \) is continuous on \([-1, 4] \).

33. \( f(x) = x^2 - 2x + 1 \) is continuous for all real \( x \).
35. \( f(x) = 3x - \cos x \) is continuous for all real \( x \).

37. \( f(x) = \frac{x}{x^2 - x} \) is not continuous at \( x = 0, 1 \). Since 
\[
\frac{x}{x^2 - x} = \frac{1}{x-1} \quad \text{for} \ x \neq 0, \ x = 0 \text{is a removable discontinuity, whereas} \ x = 1 \text{is a nonremovable discontinuity.}
\]

39. \( f(x) = \frac{x}{x^2 + 1} \) is continuous for all real \( x \).

41. \( f(x) = \frac{x + 2}{(x + 2)(x - 5)} \) has a nonremovable discontinuity at \( x = 5 \) since \( \lim_{x \to 5} f(x) \) does not exist, and has a removable discontinuity at \( x = -2 \) since 
\[
\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.
\]

43. \( f(x) = \frac{|x + 2|}{x + 2} \) has a nonremovable discontinuity at \( x = -2 \) since \( \lim_{x \to -2} f(x) \) does not exist.

45. \( f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases} \) has a possible discontinuity at \( x = 1 \).

1. \( f(1) = 1 \)

2. \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1 \) \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 \) \( \lim_{x \to 1} f(x) = 1 \)

3. \( f(1) = \lim_{x \to 1} f(x) \)

\( f \) is continuous at \( x = 1 \), therefore, \( f \) is continuous for all real \( x \).

47. \( f(x) = \begin{cases} \frac{x + 1}{2} + 1, & x \leq 2 \\ \frac{3 - x}{x}, & x > 2 \end{cases} \) has a possible discontinuity at \( x = 2 \).

1. \( f(2) = \frac{2}{2} + 1 = 2 \)

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left( \frac{x + 1}{2} \right) = 2
\]

2. \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{3 - x}{x} = 1 \) \( \lim_{x \to 2} f(x) \) does not exist.

Therefore, \( f \) has a nonremovable discontinuity at \( x = 2 \).

49. \( f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ \frac{\pi x}{4}, & |x| \geq 1 \end{cases} \) has possible discontinuities at \( x = -1, x = 1 \).

1. \( f(-1) = -1 \) \( f(1) = 1 \)

2. \( \lim_{x \to -1} f(x) = -1 \) \( \lim_{x \to 1} f(x) = 1 \)

3. \( f(-1) = \lim_{x \to -1} f(x) \) \( f(1) = \lim_{x \to -1} f(x) \)

\( f \) is continuous at \( x = \pm 1 \), therefore, \( f \) is continuous for all real \( x \).
51. \( f(x) = \csc 2x \) has nonremovable discontinuities at integer multiples of \( \pi/2 \).

55. \( \lim_{x \to 0^+} f(x) = 0 \)
\( \lim_{x \to 0^-} f(x) = 0 \)
\( f \) is not continuous at \( x = -2 \).

53. \( f(x) = \lceil x - 1 \rceil \) has nonremovable discontinuities at each integer \( k \).

57. \( f(2) = 8 \)
Find \( a \) so that \( \lim_{x \to 2^-} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2. \)

59. Find \( a \) and \( b \) such that \( \lim_{x \to 1^+} (ax + b) = -a + b = 2 \) and \( \lim_{x \to 3^-} (ax + b) = 3a + b = -2. \)
\[
\begin{align*}
a - b &= -2 \\
(+) 3a + b &= -2 \\
4a &= -4 \\
a &= -1 \\
b &= 2 + (-1) = 1 \end{align*}
\]

61. \( f(g(x)) = (x - 1)^2 \)
Continuous for all real \( x \).

65. \( y = \lfloor x \rfloor - x \)
Nonremovable discontinuity at each integer

63. \( f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1} \)
Nonremovable discontinuities at \( x = \pm 1 \)

67. \( f(x) = \begin{cases} 
2x - 4, & x \leq 3 \\
\frac{x^2}{x^2 - 3}, & x > 3 
\end{cases} \)
Nonremovable discontinuity at \( x = 3 \)

69. \( f(x) = \frac{x}{x^2 + 1} \)
Continuous on \( (-\infty, \infty) \)

71. \( f(x) = \sec \frac{\pi x}{4} \)
Continuous on:
\[ \ldots \ldots (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots \ldots \]

75. \( f(x) = \frac{1}{100} x^4 - x^3 + 3 \) is continuous on \([1, 2]\).
\( f(1) = \frac{13}{10} \) and \( f(2) = -4. \) By the Intermediate Value Theorem, \( f(c) = 0 \) for at least one value of \( c \) between 1 and 2.

The graph appears to be continuous on the interval \([-4, 4]\). Since \( f(0) \) is not defined, we know that \( f \) has a discontinuity at \( x = 0 \). This discontinuity is removable so it does not show up on the graph.
77. \( f(x) = x^2 - 2 - \cos x \) is continuous on \([0, \pi]\).

\[ f(0) = -3 \text{ and } f(\pi) = \pi^2 - 1 > 0. \]

By the Intermediate Value Theorem, \( f(c) = 0 \) for the least one value of \( c \) between 0 and \( \pi \).

81. \( g(t) = 2 \cos t - 3t \)

\( g \) is continuous on \([0, 1]\).

\[ g(0) = 2 > 0 \text{ and } g(1) = -1.9 < 0. \]

By the Intermediate Value Theorem, \( g(t) = 0 \) for at least one value \( t \) between 0 and 1. Using a graphing utility, we find that \( t = 0.5636 \).

83. \( f(x) = x^2 + x - 1 \)

\( f \) is continuous on \([0, 5]\).

\[ f(0) = -1 \text{ and } f(5) = 29 \]

\(-1 < 11 < 29 \)

The Intermediate Value Theorem applies.

\[ x^2 + x - 1 = 11 \]

\[ x^2 + x - 12 = 0 \]

\[ (x + 4)(x - 3) = 0 \]

\[ x = -4 \text{ or } x = 3 \]

\[ c = 3 \text{ (} x = -4 \text{ is not in the interval.)} \]

Thus, \( f(3) = 11 \).

87. (a) The limit does not exist at \( x = c \).

(b) The function is not defined at \( x = c \).

(c) The limit exists at \( x = c \), but it is not equal to the value of the function at \( x = c \).

(d) The limit does not exist at \( x = c \).

89. \begin{align*}
\text{The function is not continuous at } x = 3 \text{ because } \\
\lim_{x \to 3^-} f(x) = 1 \neq 0 = \lim_{x \to 3^+} f(x). 
\end{align*}

91. The functions agree for integer values of \( x \):

\[ g(x) = 3 - \lfloor -x \rfloor = 3 \quad \text{and} \quad f(x) = 3 + \lfloor x \rfloor = 3 + x \quad \text{for } x \text{ an integer} \]

However, for non-integer values of \( x \), the functions differ by 1.

\[ f(x) = 3 + \lfloor x \rfloor = g(x) - 1 = 2 - \lfloor -x \rfloor. \]

For example, \( f(\frac{3}{2}) = 3 + 0 = 3, g(\frac{3}{2}) = 3 - (-1) = 4. \)
93. \( N(t) = 25 \left( \lfloor \frac{t + 2}{2} \rfloor - t \right) \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>1.8</th>
<th>2</th>
<th>3</th>
<th>3.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N(t) )</td>
<td>50</td>
<td>25</td>
<td>5</td>
<td>50</td>
<td>25</td>
<td>5</td>
</tr>
</tbody>
</table>

Discontinuous at every positive even integer. The company replenishes its inventory every two months.

95. Let \( V = \frac{4}{3} \pi r^3 \) be the volume of a sphere of radius \( r \)

\[
V(1) = \frac{4}{3} \pi \approx 4.19 \\
V(5) = \frac{4}{3} \pi (5^3) \approx 523.6
\]

Since \( 4.19 < 275 < 523.6 \), the Intermediate Value Theorem implies that there is at least one value \( r \) between 1 and 5 such that \( V(r) = 275 \). (In fact, \( r \approx 4.0341 \).)

97. Let \( c \) be any real number. Then \( \lim_{x \to c} f(x) \) does not exist since there are both rational and irrational numbers arbitrarily close to \( c \). Therefore, \( f \) is not continuous at \( c \).

99. \( \text{sgn}(x) = \begin{cases} 
-1, & \text{if } x < 0 \\
0, & \text{if } x = 0 \\
1, & \text{if } x > 0 
\end{cases} \)

(a) \( \lim_{x \to 0^-} \text{sgn}(x) = -1 \)
(b) \( \lim_{x \to 0^+} \text{sgn}(x) = 1 \)
(c) \( \lim_{x \to 0} \text{sgn}(x) \) does not exist.

101. True; if \( f(x) = g(x) \), \( x \neq c \), then \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) \) and at least one of these limits (if they exist) does not equal the corresponding function at \( x = c \).

103. False; \( f(1) \) is not defined and \( \lim_{x \to 1} f(x) \) does not exist.

105. (a) \( f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < b \\
b & \text{if } b \leq x \leq 2b 
\end{cases} \)

(b) \( g(x) = \begin{cases} 
\frac{x}{2} & \text{if } 0 \leq x \leq b \\
b - \frac{x}{2} & \text{if } b \leq x \leq 2b 
\end{cases} \)

Continuous on \([0, 2b]\).
107. \( f(x) = \frac{\sqrt{x + c^2} - c}{x} \), \( c > 0 \)

Domain: \( x + c^2 \geq 0 \implies x \geq -c^2 \) and \( x \neq 0 \), \([-c^2, 0) \cup (0, \infty)\)

\[
\lim_{x \to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x \to 0} \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} \cdot \frac{(x + c^2) - c^2}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}
\]

Define \( f(0) = 1/(2c) \) to make \( f \) continuous at \( x = 0 \).

109. \( h(x) = \lfloor x \rfloor \)

\( h \) has nonremovable discontinuities at \( x = \pm 1, \pm 2, \pm 3, \ldots \).

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Section 1.5 Infinite Limits

1. \( \lim_{x \to 2^{-}} 2 \left| \frac{x}{x^2 - 4} \right| = \infty \)

2. \( \lim_{x \to -2^{-}} 2 \left| \frac{x}{x^2 - 4} \right| = \infty \)

3. \( \lim_{x \to \pi^{-}} \tan \frac{\pi x}{4} = -\infty \)

4. \( \lim_{x \to \pi^{-}} \tan \frac{\pi x}{4} = \infty \)

5. \( f(x) = \frac{1}{x^2 - 9} \)

\[
\begin{array}{cccccccc}
 x & -3.5 & -3.1 & -3.01 & -3.001 & -2.999 & -2.99 & -2.9 & -2.5 \\
 f(x) & 0.308 & 1.639 & 16.64 & 166.6 & -166.7 & -16.69 & -1.695 & -0.364 \\
\end{array}
\]

\( \lim_{x \to -3} f(x) = \infty \)

\( \lim_{x \to -3} f(x) = -\infty \)

7. \( f(x) = \frac{x^2}{x^2 - 9} \)

\[
\begin{array}{cccccccc}
 x & -3.5 & -3.1 & -3.01 & -3.001 & -2.999 & -2.99 & -2.9 & -2.5 \\
 f(x) & 3.769 & 15.75 & 150.8 & 1501 & -1499 & -149.3 & -14.25 & -2.273 \\
\end{array}
\]

\( \lim_{x \to -3} f(x) = \infty \)

\( \lim_{x \to -3} f(x) = -\infty \)
9. \( \lim_{x \to 0} \frac{1}{x} = \infty = \lim_{x \to 0} \frac{1}{x} \)

Therefore, \( x = 0 \) is a vertical asymptote.

13. \( \lim_{x \to -2} \frac{x^2}{x^2 - 4} = \infty \) and \( \lim_{x \to -2} \frac{x^2}{x^2 - 4} = -\infty \)

Therefore, \( x = -2 \) is a vertical asymptote.

17. \( f(x) = \tan 2x = \frac{\sin 2x}{\cos 2x} \) has vertical asymptotes at

\[ x = \frac{(2n + 1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2}, \text{ } n \text{ any integer.} \]

21. \( \lim_{x \to -2} \frac{x}{(x + 2)(x - 1)} = \infty \)

\( \lim_{x \to -2} \frac{x}{(x + 2)(x - 1)} = -\infty \)

Therefore, \( x = -2 \) is a vertical asymptote.

\( \lim_{x \to -1} \frac{x}{(x + 2)(x - 1)} = \infty \)

\( \lim_{x \to -1} \frac{x}{(x + 2)(x - 1)} = -\infty \)

Therefore, \( x = 1 \) is a vertical asymptote.

25. \( f(x) = \frac{(x - 5)(x + 3)}{(x - 5)(x^2 + 1)} = \frac{x + 3}{x^2 + 1}, \text{ } x \neq 5 \)

No vertical asymptotes. The graph has a hole at \( x = 5 \).

11. \( \lim_{x \to 2} \frac{x^2 - 2}{(x - 2)(x + 1)} = \infty \)

\( \lim_{x \to 2} \frac{x^2 - 2}{(x + 1)(x - 2)} = -\infty \)

Therefore, \( x = 2 \) is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

19. \( \lim_{t \to 0} \left( 1 - \frac{4}{t^2} \right) = -\infty = \lim_{t \to 0} \left( 1 - \frac{4}{t^2} \right) \)

Therefore, \( t = 0 \) is a vertical asymptote.

23. \( f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x + 1)(x^2 - x + 1)}{x + 1} \)

has no vertical asymptote since

\( \lim_{x \to -1} f(x) = \lim_{x \to -1} (x^2 - x + 1) = 3 \)

27. \( s(t) = \frac{t}{\sin t} \) has vertical asymptotes at \( t = n\pi, \text{ } n \text{ a nonzero integer. There is no vertical asymptote at } t = 0 \text{ since} \)

\( \lim_{t \to 0} \frac{t}{\sin t} = 1. \)
29. \[ \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} (x - 1) = -2 \]

Removable discontinuity at \( x = -1 \)

31. \[ \lim_{x \to -1} \frac{x^2 + 1}{x + 1} = \infty \]

Vertical asymptote at \( x = -1 \)

33. \[ \lim_{x \to 2} \frac{x - 3}{x - 2} = -\infty \]

35. \[ \lim_{x \to 3} \frac{x^2}{(x - 3)(x + 3)} = \infty \]

37. \[ \lim_{x \to -3} \frac{x^2 + 2x - 3}{x^2 + x - 6} = \lim_{x \to -3} \frac{x - 1}{x - 2} = \frac{4}{5} \]

39. \[ \lim_{x \to 1} \frac{x^2 - x}{(x^2 + 1)(x - 1)} = \lim_{x \to 1} \frac{x}{x^2 + 1} = \frac{1}{2} \]

41. \[ \lim_{x \to 0^+} \left( 1 + \frac{1}{x} \right) = -\infty \]

43. \[ \lim_{x \to 0^+} \sin x = \infty \]

45. \[ \lim_{x \to \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x \to \pi} \left( \sqrt{x} \sin x \right) = 0 \]

47. \[ \lim_{x \to \pi} x \sec(\pi x) = \infty \quad \text{and} \quad \lim_{x \to (1/2)^-} x \sec(\pi x) = -\infty. \]

Therefore, \( \lim_{x \to (1/2)^-} x \sec(\pi x) \) does not exist.

49. \( f(x) = \frac{x^2 + x + 1}{x^3 - 1} \)

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x - 1} = \infty \]

51. \( f(x) = \frac{1}{x^2 - 25} \)

\[ \lim_{x \to 5} f(x) = -\infty \]

53. A limit in which \( f(x) \) increases or decreases without bound as \( x \) approaches \( c \) is called an infinite limit. \( \infty \) is not a number. Rather, the symbol

\[ \lim_{x \to c} f(x) = \infty \]

says how the limit fails to exist.

55. One answer is \( f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12} \)

57. \( S = \frac{k}{1 - r}, \quad 0 < |r| < 1. \) Assume \( k \neq 0. \)

\[ \lim_{r \to 1} S = \lim_{r \to 1} \frac{k}{1 - r} = \infty \quad \text{(or} -\infty \text{if} \ k \ < \ 0) \]
61. \( C = \frac{528x}{100 - x}, \quad 0 \leq x < 100 \)

(a) \( C(25) = 176 \text{ million} \)

(b) \( C(50) = 528 \text{ million} \)

(c) \( C(75) = 1584 \text{ million} \)

(d) \( \lim_{x \to 100} \frac{528}{100 - x} = \infty \) Thus, it is not possible.

63. (a) \( r = \frac{2(7)}{\sqrt{625 - 49}} = \frac{7}{12} \text{ ft/sec} \)

(b) \( r = \frac{2(15)}{\sqrt{625 - 225}} = \frac{3}{2} \text{ ft/sec} \)

(c) \( \lim_{x \to 25} \frac{2x}{\sqrt{625 - x^2}} = \infty \)

65. (a) \[
\begin{array}{cccccccc}
   x & 1 & 0.5 & 0.2 & 0.1 & 0.01 & 0.001 & 0.0001 \\
   f(x) & 0.1585 & 0.0411 & 0.0067 & 0.0017 & \approx 0 & \approx 0 & \approx 0 \\
\end{array}
\]

\[
\lim_{x \to 0^+} \frac{x - \sin x}{x} = 0
\]

(b) \[
\begin{array}{cccccccc}
   x & 1 & 0.5 & 0.2 & 0.1 & 0.01 & 0.001 & 0.0001 \\
   f(x) & 0.1585 & 0.0823 & 0.0333 & 0.0167 & 0.0017 & \approx 0 & \approx 0 \\
\end{array}
\]

\[
\lim_{x \to 0^+} \frac{x - \sin x}{x^2} = 0
\]

(c) \[
\begin{array}{cccccccc}
   x & 1 & 0.5 & 0.2 & 0.1 & 0.01 & 0.001 & 0.0001 \\
   f(x) & 0.1585 & 0.1646 & 0.1663 & 0.1666 & 0.1667 & 0.1667 & 0.1667 \\
\end{array}
\]

\[
\lim_{x \to 0^+} \frac{x - \sin x}{x^3} = 0.1167 \ (1/6)
\]

(d) \[
\begin{array}{cccccccc}
   x & 1 & 0.5 & 0.2 & 0.1 & 0.01 & 0.001 & 0.0001 \\
   f(x) & 0.1585 & 0.3292 & 0.8317 & 1.6658 & 16.67 & 166.7 & 1667.0 \\
\end{array}
\]

\[
\lim_{x \to 0^+} \frac{x - \sin x}{x^4} = \infty
\]

For \( n \geq 3 \), \( \lim_{x \to 0^+} \frac{x - \sin x}{x^n} = \infty \).
67. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes \(1700/2 = 850\) revolutions per minute.

(c) \(2(20 \cot \phi) + 2(10 \cot \phi)\): straight sections.

The angle subtended in each circle is

\[
2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi.
\]

Thus, the length of the belt around the pulleys is

\[20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi).
\]

Total length = 60 cot \(\phi + 30(\pi + 2\phi)

Domain: \((0, \frac{\pi}{2})\)

69. False; for instance, let

\[
f(x) = \frac{x^2 - 1}{x - 1} \text{ or } g(x) = \frac{x}{x^2 + 1}.
\]

71. False; let

\[
f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ \frac{3}{x}, & x = 0. \end{cases}
\]

The graph of \(f\) has a vertical asymptote at \(x = 0\), but \(f(0) = 3\).

73. Given \(\lim \limits_{x \to a} f(x) = \infty\) and \(\lim \limits_{x \to a} g(x) = L\):

(2) Product:

If \(L > 0\), then for \(\epsilon = L/2 > 0\) there exists \(\delta_1 > 0\) such that \(|g(x) - L| < L/2\) whenever \(0 < |x - c| < \delta_1\). Thus, \(L/2 < g(x) < 3L/2\). Since \(\lim \limits_{x \to a} f(x) = \infty\) then for \(M > 0\), there exists \(\delta_2 > 0\) such that \(f(x) > M(2/L)\) whenever \(0 < |x - c| < \delta_2\). Let \(\delta\) be the smaller of \(\delta_1\) and \(\delta_2\). Then for \(0 < |x - c| < \delta\), we have \(f(x)g(x) > M(2/L)(L/2) = M\).

Therefore \(\lim \limits_{x \to a} f(x)g(x) = \infty\). The proof is similar for \(L < 0\).

(3) Quotient: Let \(\epsilon > 0\) be given.

There exists \(\delta_1 > 0\) such that \(f(x) > 3L/2\epsilon\) whenever \(0 < |x - c| < \delta_1\) and there exists \(\delta_2 > 0\) such that \(|g(x) - L| < L/2\) whenever \(0 < |x - c| < \delta_2\). This inequality gives us \(L/2 < g(x) < 3L/2\). Let \(\delta\) be the smaller of \(\delta_1\) and \(\delta_2\). Then for \(0 < |x - c| < \delta\), we have

\[
\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\epsilon} = \epsilon.
\]

Therefore, \(\lim \limits_{x \to a} \frac{g(x)}{f(x)} = 0\).

75. Given \(\lim \limits_{x \to a} \frac{1}{f(x)} = 0\).

Suppose \(\lim \limits_{x \to a} f(x)\) exists and equals \(L\). Then,

\[
\lim \limits_{x \to a} \frac{1}{f(x)} = \lim \limits_{x \to a} \frac{1}{\lim \limits_{x \to a} f(x)} = \frac{1}{L} = 0.
\]

This is not possible. Thus, \(\lim \limits_{x \to a} f(x)\) does not exist.