83. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b (x^2+1)^{-4}(2x\,dx) = \lim_{b\to\infty}\left[-\frac{1}{6}\frac{1}{(x^2+1)^3}\right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2+1)^{n+3}}\,dx = \lim_{b\to\infty}\left[\frac{-x^{2n-2}}{2(n+2)(x^2+1)^n} + 2\right]_0^b + \frac{n-1}{n+2}\int_0^\infty \frac{x^{2n-3}}{(x^2+1)^{n+2}}\,dx = 0 + \frac{n-1}{n+2}(I_{n-1})$$

$u = x^{2n-2}$, $du = (2n-2)x^{2n-3}\,dx$, $dv = \dfrac{x}{(x^2+1)^{n+3}}\,dx$, $v = \dfrac{-1}{2(n+2)(x^2+1)^{n+2}}$

(a) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{6(x^2+1)^3}\right]_0^b = \frac{1}{6}$

(b) $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{1}{4}\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}$

(c) $\displaystyle\int_0^\infty \frac{x^5}{(x^2+1)^6} = \frac{2}{5}\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{2}{5}\left(\frac{1}{24}\right) = \frac{1}{60}$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\displaystyle\lim_{x\to\infty} 1/(x+1) = 0$, but $\displaystyle\int_0^\infty \frac{1}{x+1}\,dx = \lim_{b\to\infty}\Big[\ln|x+1|\Big]_0^b = \infty$.

Diverges

87. True

Review Exercises for Chapter 7

1. $$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int (x^2-1)^{1/2}(2x)\,dx = \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C$$

3. $$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1| + C$$

5. $$\int \frac{\ln(2x)}{x}\,dx = \frac{(\ln 2x)^2}{2} + C$$

7. $$\int \frac{16}{\sqrt{16-x^2}}\,dx = 16\arcsin\left(\frac{x}{4}\right) + C$$

9. $$\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)$$

$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

(1) $dv = \sin 3x\,dx \implies v = -\dfrac{1}{3}\cos 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \implies v = \dfrac{1}{3}\sin 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

11. $u = x,\ du = dx,\ dv = (x-5)^{1/2}\,dx,\ v = \frac{2}{3}(x-5)^{3/2}$

$$\int x\sqrt{x-5}\,dx = \frac{2}{3}x(x-5)^{3/2} - \int \frac{2}{3}(x-5)^{3/2}\,dx$$
$$= \frac{2}{3}x(x-5)^{3/2} - \frac{4}{15}(x-5)^{5/2} + C$$
$$= (x-5)^{3/2}\left[\frac{2}{3}x - \frac{4}{15}(x-5)\right] + C$$
$$= (x-5)^{3/2}\left[\frac{6}{15}x + \frac{4}{3}\right] + C$$
$$= \frac{2}{15}(x-5)^{3/2}[3x+10] + C$$

13. $$\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$$
$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$
$$= -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(1) $dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = \cos 2x\,dx \implies v = \frac{1}{2}\sin 2x$

$u = x \implies du = dx$

15. $$\int x\arcsin 2x\,dx = \frac{x^2}{2}\arcsin 2x - \int \frac{x^2}{\sqrt{1-4x^2}}\,dx$$
$$= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\int \frac{2(2x)^2}{\sqrt{1-(2x)^2}}\,dx$$
$$= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\left(\frac{1}{2}\right)\left[-(2x)\sqrt{1-4x^2} + \arcsin 2x\right] + C \quad \text{(by Formula 43 of Integration Tables)}$$
$$= \frac{1}{16}\left[(8x^2-1)\arcsin 2x + 2x\sqrt{1-4x^2}\right] + C$$

$dv = x\,dx \implies v = \frac{x^2}{2}$

$u = \arcsin 2x \implies du = \frac{2}{\sqrt{1-4x^2}}\,dx$

17. $$\int \cos^3(\pi x - 1)\,dx = \int [1 - \sin^2(\pi x - 1)]\cos(\pi x - 1)\,dx$$
$$= \frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{1}{3}\sin^3(\pi x - 1)\right] + C$$
$$= \frac{1}{3\pi}\sin(\pi x - 1)[3 - \sin^2(\pi x - 1)] + C$$
$$= \frac{1}{3\pi}\sin(\pi x - 1)[3 - (1 - \cos^2(\pi x - 1))] + C$$
$$= \frac{1}{3\pi}\sin(\pi x - 1)[2 + \cos^2(\pi x - 1)] + C$$

19. $$\int \sec^4\left(\frac{x}{2}\right)dx = \int\left[\tan^2\left(\frac{x}{2}\right) + 1\right]\sec^2\left(\frac{x}{2}\right)dx$$
$$= \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx + \int \sec^2\left(\frac{x}{2}\right)dx$$
$$= \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + C = \frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C$$

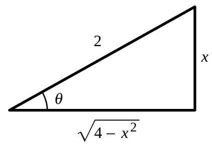
21. $$\int \frac{1}{1-\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^2\theta}\,d\theta = \int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$$

23. $\displaystyle\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = \int \frac{-24\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)}$

$$= -3\int \csc^2\theta\,d\theta$$

$$= 3\cot\theta + C$$

$$= \frac{3\sqrt{4-x^2}}{x} + C$$

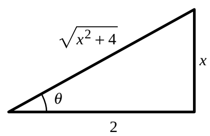


$x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$

25. $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$4 + x^2 = 4\sec^2\theta$



$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta$$

$$= 8\int \tan^3\theta\sec\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

$$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 8\left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right] + C$$

$$= \sqrt{x^2+4}\left[\frac{1}{3}(x^2+4) - 4\right] + C$$

$$= \frac{1}{3}x^2\sqrt{x^2+4} - \frac{8}{3}\sqrt{x^2+4} + C$$

$$= \frac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$$

27. $\displaystyle\int \sqrt{4-x^2}\,dx = \int (2\cos\theta)(2\cos\theta)\,d\theta$

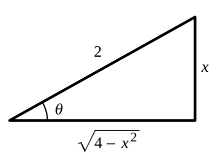
$$= 2\int (1 + \cos 2\theta)\,d\theta$$

$$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= 2(\theta + \sin\theta\cos\theta) + C$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right) + \frac{x}{2}\left(\frac{\sqrt{4-x^2}}{2}\right)\right] + C$$

$$= \frac{1}{2}\left[4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2}\right] + C$$



$x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$

29. (a) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = 8\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta$

$$= 8\int (\cos^{-4}\theta - \cos^{-2}\theta)\sin\theta\,d\theta$$

$$= \frac{8}{3}\sec\theta(\sec^2\theta - 3) + C$$

$$= \frac{\sqrt{4+x^2}}{3}(x^2 - 8) + C$$

$x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta$

(b) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int (u^2 - 4)\,du$

$$= \frac{1}{3}u^3 - 4u + C$$

$$= \frac{u}{3}(u^2 - 12) + C$$

$$= \frac{\sqrt{4+x^2}}{3}(x^2 - 8) + C$$

$u^2 = 4 + x^2,\ 2u\,du = 2x\,dx$

(c) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3}(x^2 - 8) + C$$

$$dv = \frac{x}{\sqrt{4+x^2}}\,dx \implies v = \sqrt{4+x^2}$$

$$u = x^2 \implies du = 2x\,dx$$

31. $\displaystyle\frac{x - 28}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2}$

$$x - 28 = A(x + 2) + B(x - 3)$$

$x = -2 \implies -30 = B(-5) \implies B = 6$

$x = 3 \implies -25 = A(5) \implies A = -5$

$$\int \frac{x - 28}{x^2 - 6 - 6}\,dx = \int\left(\frac{-5}{x - 3} + \frac{6}{x + 2}\right)dx = -5\ln|x - 3| + 6\ln|x + 2| + C$$

33. $\displaystyle\frac{x^2 + 2x}{(x - 1)(x^2 + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 1}$

$$x^2 + 2x = A(x^2 + 1) + (Bx + C)(x - 1)$$

Let $x = 1$: $3 = 2A \implies A = \dfrac{3}{2}$

Let $x = 0$: $0 = A - C \implies C = \dfrac{3}{2}$

Let $x = 2$: $8 = 5A + 2B + C \implies B = -\dfrac{1}{2}$

$$\int \frac{x^2 + 2x}{x^3 - x^2 + x - 1}\,dx = \frac{3}{2}\int \frac{1}{x - 1}\,dx - \frac{1}{2}\int \frac{x - 3}{x^2 + 1}\,dx$$

$$= \frac{3}{2}\int \frac{1}{x - 1}\,dx - \frac{1}{4}\int \frac{2x}{x^2 + 1}\,dx + \frac{3}{2}\int \frac{1}{x^2 + 1}\,dx$$

$$= \frac{3}{2}\ln|x - 1| - \frac{1}{4}\ln|x^2 + 1| + \frac{3}{2}\arctan x + C$$

$$= \frac{1}{4}[6\ln|x - 1| - \ln(x^2 + 1) + 6\arctan x] + C$$

35. $\dfrac{x^2}{x^2+2x-15} = 1 + \dfrac{15-2x}{x^2+2x-15}$

$$\frac{15-2x}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

$$15 - 2x = A(x+5) + B(x-3)$$

Let $x = 3$: $9 = 8A \implies A = \dfrac{9}{8}$

Let $x = -5$: $25 = -8B \implies B = -\dfrac{25}{8}$

$$\int \frac{x^2}{x^2+2x-15}\,dx = \int dx + \frac{9}{8}\int \frac{1}{x-3}\,dx - \frac{25}{8}\int \frac{1}{x+5}\,dx$$

$$= x + \frac{9}{8}\ln|x-3| - \frac{25}{8}\ln|x+5| + C$$

37. $\displaystyle\int \frac{x}{(2+3x)^2}\,dx = \frac{1}{9}\left[\frac{2}{2+3x} + \ln|2+3x|\right] + C$

(Formula 4)

39. $\displaystyle\int \frac{x}{1+\sin x^2}\,dx = \frac{1}{2}\int \frac{1}{1+\sin u}\,du$ $\quad (u = x^2)$

$$= \frac{1}{2}[\tan u - \sec u] + C \qquad \text{(Formula 56)}$$

$$= \frac{1}{2}[\tan x^2 - \sec x^2] + C$$

41. $\displaystyle\int \frac{x}{x^2+4x+8}\,dx = \frac{1}{2}\left[\ln|x^2+4x+8| - 4\int \frac{1}{x^2+4x+8}\,dx\right]$ $\quad$ (Formula 15)

$$= \frac{1}{2}[\ln|x^2+4x+8|] - 2\left[\frac{2}{\sqrt{32-16}}\arctan\left(\frac{2x+4}{\sqrt{32-16}}\right)\right] + C \quad \text{(Formula 14)}$$

$$= \frac{1}{2}\ln|x^2+4x+8| - \arctan\left(1 + \frac{x}{2}\right) + C$$

43. $\displaystyle\int \frac{1}{\sin \pi x \cos \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{\sin \pi x \cos \pi x}(\pi)\,dx$ $\quad (u = \pi x)$

$$= \frac{1}{\pi}\ln|\tan \pi x| + C \qquad \text{(Formula 58)}$$

45. $dv = dx \implies v = x$

$u = (\ln x)^n \implies du = n(\ln x)^{n-1}\dfrac{1}{x}dx$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

47. $\displaystyle\int \theta \sin\theta \cos\theta\,d\theta = \frac{1}{2}\int \theta \sin 2\theta\,d\theta$

$$= -\frac{1}{4}\theta\cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$$

$dv = \sin 2\theta\,d\theta \implies v = -\dfrac{1}{2}\cos 2\theta$

$u = \theta \implies du = d\theta$

49. $$\int \frac{x^{1/4}}{1+x^{1/2}}\,dx = 4\int \frac{u(u^3)}{1+u^2}\,du$$
$$= 4\int \left(u^2 - 1 + \frac{1}{u^2+1}\right) du$$
$$= 4\left(\frac{1}{3}u^3 - u + \arctan u\right) + C$$
$$= \frac{4}{3}\left[x^{3/4} - 3x^{1/4} + 3\arctan(x^{1/4})\right] + C$$

$y = \sqrt[4]{x}$, $x = u^4$, $dx = 4u^3\,du$

51. $$\int \sqrt{1+\cos x}\,dx = \int \frac{\sin x}{\sqrt{1-\cos x}}\,dx$$
$$= \int (1-\cos x)^{-1/2}(\sin x)\,dx$$
$$= 2\sqrt{1-\cos x} + C$$

$u = 1 - \cos x$, $du = \sin x\,dx$

53. $$\int \cos x \ln(\sin x)\,dx = \sin x \ln(\sin x) - \int \cos x\,dx$$
$$= \sin x \ln(\sin x) - \sin x + C$$

$dv = \cos x\,dx \implies v = \sin x$

$u = \ln(\sin x) \implies du = \dfrac{\cos x}{\sin x}\,dx$

55. $$y = \int \frac{9}{x^2-9}\,dx = \frac{3}{2}\ln\left|\frac{x-3}{x+3}\right| + C$$

(by Formula 24 of Integration Tables)

57. $$y = \int \ln(x^2+x)dx = x\ln|x^2+x| - \int \frac{2x^2+x}{x^2+x}\,dx$$
$$= x\ln|x^2+x| - \int \frac{2x+1}{x+1}\,dx$$
$$= x\ln|x^2+x| - \int 2dx + \int \frac{1}{x+1}\,dx$$
$$= x\ln|x^2+x| - 2x + \ln|x+1| + C$$

$dv = dx \implies v = x$

$u = \ln(x^2+x) \implies du = \dfrac{2x+1}{x^2+x}\,dx$

59. $$\int_2^{\sqrt{5}} x(x^2-4)^{3/2}\,dx = \left[\frac{1}{5}(x^2-4)^{5/2}\right]_2^{\sqrt{5}} = \frac{1}{5}$$

61. $$\int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}(\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$$

63. $$\int_0^{\pi} x\sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^{\pi} = \pi$$

65. $$A = \int_0^4 x\sqrt{4-x}\,dx = \int_2^0 (4-u^2)u(-2u)\,du$$
$$= \int_2^0 2(u^4 - 4u^2)\,du$$
$$= \left[2\left(\frac{u^5}{5} - \frac{4u^3}{3}\right)\right]_2^0 = \frac{128}{15}$$

$u = \sqrt{4-x}$, $x = 4 - u^2$, $dx = -2u\,du$

67. By symmetry, $\bar{x} = 0$, $A = \dfrac{1}{2}\pi$.

$$\bar{y} = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_{-1}^{1}\left(\sqrt{1-x^2}\right)^2 dx = \frac{1}{\pi}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$$

69. $$s = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx \approx 3.82$$

71. $$\lim_{x\to 1}\left[\frac{(\ln x)^2}{x-1}\right] = \lim_{x\to 1}\left[\frac{2(1/x)\ln x}{1}\right] = 0$$

73. $$\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{4e^{2x}}{2} = \infty$$

75. $$y = \lim_{x\to\infty}(\ln x)^{2/x}$$

$$\ln y = \lim_{x\to\infty}\frac{2\ln(\ln x)}{x} = \lim_{x\to\infty}\left[\frac{2/(x\ln x)}{1}\right] = 0$$

Since $\ln y = 0$, $y = 1$.

77. $\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n = 1000 \lim_{n\to\infty}\left(1 + \frac{0.09}{n}\right)^n$

Let $y = \lim_{n\to\infty}\left(1 + \frac{0.09}{n}\right)^n$.

$$\ln y = \lim_{n\to\infty} n\ln\left(1 + \frac{0.09}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1 + \frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{0.09}{1 + \left(\frac{0.09}{n}\right)} = 0.09$$

Thus, $\ln y = 0.09 \implies y = e^{0.09}$ and $\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17$.

79. $\int_0^{16}\frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_b^{16} = \frac{32}{3}$

Converges

81. $\int_1^{\infty} x^2\ln x\,dx = \lim_{b\to\infty}\left[\frac{x^3}{9}(-1 + 3\ln x)\right]_1^b = \infty$

Diverges

83. $\int_0^{t_0} 500{,}000e^{-0.05t}\,dt = \left[\frac{500{,}000}{-0.05}e^{-0.05t}\right]_0^{t_0}$

$$= \frac{-500{,}000}{0.05}\left(e^{-0.05t_0} - 1\right)$$

$$= 10{,}000{,}000\left(1 - e^{-0.05t_0}\right)$$

(a) $t_0 = 20$: \$6,321,205.59

(b) $t_0 \to \infty$: \$10,000,000

85. (a) $P(13 \le x < \infty) = \frac{1}{0.95\sqrt{2\pi}}\int_{13}^{\infty} e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.4581$

(b) $P(15 \le x < 20) = \frac{1}{0.95\sqrt{2\pi}}\int_{15}^{\infty} e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.0135$

Problem Solving for Chapter 7

1. (a) $\int_{-1}^{1}(1 - x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$

$$\int_{-1}^{1}(1 - x^2)^2\,dx = \int_{-1}^{1}(1 - 2x^2 + x^4)\,dx = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16}{15}$$

(b) Let $x = \sin u$, $dx = \cos u\,du$, $1 - x^2 = 1 - \sin^2 u = \cos^2 u$.

$$\int_{-1}^{1}(1 - x^2)^n\,dx = \int_{-\pi/2}^{\pi/2}(\cos^2 u)^n\cos u\,du$$

$$= \int_{-\pi/2}^{\pi/2}\cos^{2n+1}u\,du$$

$$= 2\left[\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{(2n)}{(2n+1)}\right] \qquad \text{(Wallis's Formula)}$$

$$= 2\left[\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}\right]$$

$$= \frac{2(2^{2n})(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

3.
$$\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 9$$
$$\lim_{x\to\infty} x\ln\left(\frac{x+c}{x-c}\right) = \ln 9$$
$$\lim_{x\to\infty}\frac{\ln(x+c)-\ln(x-c)}{1/x} = \ln 9$$
$$\lim_{x\to\infty}\frac{\dfrac{1}{x+c}-\dfrac{1}{x-c}}{-\dfrac{1}{x^2}} = \ln 9$$
$$\lim_{x\to\infty}\frac{-2c}{(x+c)(x-c)}(-x^2) = \ln 9$$
$$\lim_{x\to\infty}\left(\frac{2cx^2}{x^2-c^2}\right) = \ln 9$$
$$2c = \ln 9$$
$$2c = 2\ln 3$$
$$c = \ln 3$$

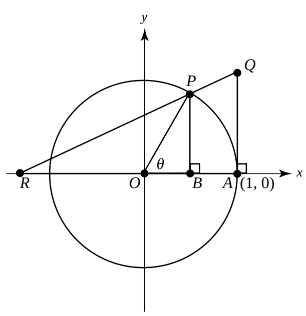
5. $\sin\theta = \dfrac{PB}{OP} = PB$, $\cos\theta = OB$

$AQ = \widehat{AP} = \theta$

$BR = OR + OB = OR + \cos\theta$

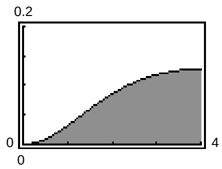
The triangles $\triangle AQR$ and $\triangle BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Longrightarrow \frac{OR+1}{\theta} = \frac{OR+\cos\theta}{\sin\theta}$$
$$\sin\theta(OR) + \sin\theta = (OR)\theta + \theta\cos\theta$$
$$OR = \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$



$$\lim_{\theta\to 0^+} OR = \lim_{\theta\to 0^+}\frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$
$$= \lim_{\theta\to 0^+}\frac{-\theta\sin\theta + \cos\theta - \cos\theta}{\cos\theta - 1}$$
$$= \lim_{\theta\to 0^+}\frac{-\theta\sin\theta}{\cos\theta - 1}$$
$$= \lim_{\theta\to 0^+}\frac{-\sin\theta - \theta\cos\theta}{-\sin\theta}$$
$$= \lim_{\theta\to 0}\frac{\cos\theta + \cos\theta - \theta\sin\theta}{\cos\theta}$$
$$= 2$$

7. (a)



Area ≈ 0.2986

(b) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $x^2 + 9 = 9\sec^2\theta$.

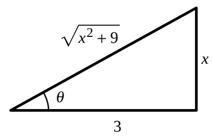
$$\int \frac{x^2}{(x^2+9)}\,dx = \int \frac{9\tan^2\theta}{(9\sec^2\theta)^{3/2}}(3\sec^2\theta\,d\theta)$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C$$



$$\text{Area} = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx = \Big[\ln\left|\sec\theta + \tan\theta\right| - \sin\theta\Big]_0^{\tan^{-1}(4/3)}$$

$$= \left[\ln\left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right) - \frac{x}{\sqrt{x^2+9}}\right]_0^4$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \frac{4}{5} = \ln 3 - \frac{4}{5}$$

(c) $x = 3\sinh u$, $dx = 3\cosh u\,du$, $x^2 + 9 = 9\sinh^2 u + 9 = 9\cosh^2 u$

$$A = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx = \int_0^{\sin^{-1}(4/3)} \frac{9\sinh^2 u}{(9\cosh^2 u)^{3/2}}(3\cosh u\,du)$$

$$= \int_0^{\sinh^{-1}(4/3)} \tanh^2 u\,du$$

$$= \int_0^{\sinh^{-1}(4/3)} (1 - \operatorname{sech}^2 u)\,du$$

$$= \Big[u - \tanh u\Big]_0^{\sinh^{-1}(4/3)}$$

$$= \sinh^{-1}\left(\frac{4}{3}\right) - \tanh\left(\sinh^{-1}\left(\frac{4}{3}\right)\right)$$

$$= \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right) - \tanh\left[\ln\left(\frac{4}{3} + \sqrt{\frac{16}{9} + 1}\right)\right]$$

$$= \ln\left(\frac{4}{3} + \frac{5}{3}\right) - \tanh\left(\ln\left(\frac{4}{3} + \frac{5}{3}\right)\right)$$

$$= \ln 3 - \tanh(\ln 3)$$

$$= \ln 3 - \frac{3 - (1/3)}{3 + (1/3)}$$

$$= \ln 3 - \frac{4}{5}$$

9. $y = \ln(1 - x^2),\ y' = \dfrac{-2x}{1 - x^2}$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

$$\begin{aligned}
\text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2}\,dx \\
&= \int_0^{1/2} \left(\frac{1 + x^2}{1 - x^2}\right) dx \\
&= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx \\
&= \int_0^{1/2} \left(-1 + \frac{1}{x + 1} + \frac{1}{1 - x}\right) dx \\
&= \Big[-x + \ln(1 + x) - \ln(1 - x)\Big]_0^{1/2} \\
&= \left(-\frac{1}{2} + \ln\frac{3}{2} - \ln\frac{1}{2}\right) \\
&= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2 \\
&= \ln 3 - \frac{1}{2} \approx 0.5986
\end{aligned}$$

11. Consider $\displaystyle\int \frac{1}{\ln x}\,dx.$

Let $u = \ln x,\ du = \dfrac{1}{dx}\,dx,\ x = e^u$. Then $\displaystyle\int \frac{1}{\ln x}\,dx = \int \frac{1}{u}e^u\,du = \int \frac{e^u}{u}\,du.$

If $\displaystyle\int \frac{1}{\ln x}\,dx$ were elementary, then $\displaystyle\int \frac{e^u}{u}\,du$ would be too, which is false.

Hence, $\displaystyle\int \frac{1}{\ln x}\,dx$ is not elementary.

13. $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$a = -c,\ b = d = 1,\ a = \sqrt{2}$

$$x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$$

$$\begin{aligned}
\int_0^1 \frac{1}{x^4 + 1}\,dx &= \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1}\,dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1}\,dx \\
&= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx \\
&= \frac{\sqrt{2}}{4}\Big[\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)\Big]_0^1 + \frac{\sqrt{2}}{8}\Big[\ln\left(x^2 + \sqrt{2}x + 1\right) - \ln\left(x^2 - \sqrt{2}x + 1\right)\Big]_0^1 \\
&= \frac{\sqrt{2}}{4}\left[\arctan\left(\sqrt{2} + 1\right) + \arctan\left(\sqrt{2} - 1\right)\right] + \frac{\sqrt{2}}{8}\left[\ln\left(2 + \sqrt{2}\right) - \ln\left(2 - \sqrt{2}\right)\right] - \frac{\sqrt{2}}{4}\left[\frac{\pi}{4} - \frac{\pi}{4}\right] - \frac{\sqrt{2}}{8}[0] \\
&\approx 0.5554 + 0.3116 \\
&\approx 0.8670
\end{aligned}$$

15. Using a graphing utility,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty$

(b) $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = 0$

(c) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) \approx -\frac{2}{3}.$

Analytically,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty + \infty = \infty$

(b)
$$\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x\cot x - 1}{x} = \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x}$$
$$= \lim_{x\to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{-x\sin x}{\sin x + x\cos x}$$
$$= \lim_{x\to 0^+}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.$$

(c)
$$\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \cot^2 x - \frac{1}{x^2}$$
$$= \frac{x^2\cot^2 x - 1}{x^2}$$

$$\lim_{x\to 0^+}\frac{x^2\cot^2 x - 1}{x^2} = \lim_{x\to 0^+}\frac{2x\cot^2 x - 2x^2\cot x\csc^2 x}{2x}$$
$$= \lim_{x\to 0^+}\frac{\cot^2 x - x\cot x\csc^2 x}{1}$$
$$= \lim_{x\to 0^+}\frac{\cos^2 x\sin x - x\cos x}{\sin^3 x}$$
$$= \lim_{x\to 0^+}\frac{(1 - \sin^2 x)\sin x - x\cos x}{\sin^3 x}$$
$$= \lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} - 1$$

Now, $\lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} = \lim_{x\to 0^+}\frac{\cos x - \cos x + x\sin x}{3\sin^2 x\cos x}$
$$= \lim_{x\to 0^+}\frac{x}{3\sin x\cdot\cos x}$$
$$= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)\frac{1}{3\cos x} = \frac{1}{3}.$$

Thus, $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \frac{1}{3} - 1 = -\frac{2}{3}.$

The form $0\cdot\infty$ is indeterminant.

17. $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{P_1}{x} + \dfrac{P_2}{x-1} + \dfrac{P_3}{x+4} + \dfrac{P_4}{x-3} \implies c_1 = 0,\ c_2 = 1,\ c_3 = -4,\ c_4 = 3$

$N(x) = x^3 - 3x^2 + 1$

$D'(x) = 4x^3 - 26x + 12$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

Thus, $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{1/12}{x} + \dfrac{1/10}{x-1} + \dfrac{111/140}{x+4} + \dfrac{1/42}{x-3}$.

19. By parts,

$$\begin{aligned}\int_a^b f(x)g''(x)\,dx &= \Big[f(x)g'(x)\Big]_a^b - \int f'(x)g'(x)\,dx \\ &= -\int_a^b f'(x)g'(x)\,dx \\ &= \Big[-f'(x)g(x)\Big]_a^b + \int_a^b g(x)f''(x)\,dx \\ &= \int_a^b f''(x)g(x)\,dx.\end{aligned}$$